GREEN'S FUNCTION TECHNIQUES

This is a general technique for solving inhomogeneous ordinary and partial differential equations. Schematically we can write an <u>inhomogeneous ordinary differential equation</u> in the form

$$L\, u(x) = f(x) \leftarrow \text{inhomogeneous term} \tag{1}$$

L = linear differential operator (e.g. $L = \frac{d^2}{dx^2} + \omega^2$). Before discussing the solution of (1) via Green's functions we examine another solution of (1) — via an expansion in the eigenfunctions of L. (This is also a review of the eigenvalue problem from last semester.) We begin by solving the eigenvalue problem (with appropriate boundary conditions)

$$L\, u_n(x) = \lambda_n u_n(x) \qquad [\text{no sum on } n] \tag{2}$$

For the operators of interest to us the eigenvectors $u_n(x)$ form a complete set so that we may expand $f(x)$ in (1) in terms of the $u_n(x)$:

$$f(x) = \sum_n \alpha_n u_n(x) \Leftrightarrow |f\rangle = \sum_n |u_n\rangle \underbrace{\langle u_n | f\rangle}_{\alpha_n} \tag{3}$$

$$u(x) = \sum_n \beta_n u_n(x) \Leftrightarrow |u\rangle = \sum_m |u_m\rangle \underbrace{\langle u_m | u\rangle}_{\beta_m} \tag{4}$$

Since $f(x)$ is a known function it follows that the α_n are known, but the β_m are not. From (1)-(4) we have:

$$\underbrace{L\, u(x) \equiv L\left\{ \sum_n \beta_n u_n(x) \right\}}_{} = f(x) \equiv \sum_n \alpha_n u_n(x) \tag{5}$$

$$\sum_n \beta_n (L\, u_n(x)) = \sum_n \beta_n (\lambda_n u_n(x)) \tag{6}$$

Hence (5) & (6) $\Rightarrow \sum_n (\beta_n \lambda_n - \alpha_n) u_n(x) = 0 \qquad (7)$

Since the functions $u_n(x)$ are assumed to be linearly-independent it follows that

$$\beta_n \lambda_n = \alpha_n \Rightarrow \boxed{\beta_n = \alpha_n/\lambda_n} \quad (8)$$

Since α_n and λ_n are known from the eigenvalue problem, this solves the inhomogeneous equation.

Example: Let $L = -d^2/dx^2$, and consider L acting on the function $u(x)$ satisfying the boundary conditions $u(1) = u(0) = 0$. We then want to solve the equation $Lu = f$ for an arbitrary f under these circumstances.

We start by writing

$$L u_n = \lambda_n u_n \Rightarrow -d^2/dx^2\, u_n(x) - \lambda_n u_n(x) = 0 \qquad (9)$$

$$\text{or } \left(d^2/dx^2 + \lambda_n\right) u_n(x) = 0 \Rightarrow u_n = (\text{Const}) \sin\sqrt{\lambda_n}\, x. \qquad (10)$$

The boundary condition $u(1) = 0 \Rightarrow \sqrt{\lambda_n} = n\pi$ ($n = \pm$ integer) $\Rightarrow \boxed{\lambda_n = n^2\pi^2}$ (11)
$n = 1, 2, 3$

Note that $\lambda_n = 0$ is excluded since this would give $u_n(x) \equiv 0$.

Hence an appropriately normalized solution is:

$$\boxed{u_n(x) = \sqrt{2}\, \sin n\pi x} \quad (12)$$

Given the eigenfunctions $u_n(x)$ we can expand $f(x)$ in the form

$$f(x) = \sum_n \alpha_n u_n(x) = \sqrt{2} \sum_{n=1}^{\infty} \alpha_n \sin n\pi x \qquad (13)$$

$$\alpha_n = \langle u_n | f \rangle = \int_0^1 dx\, u_n^*(x) f(x) = \sqrt{2} \int_0^1 dx\, (\sin n\pi x) f(x) \qquad (14)$$

The orthonormality condition is:

$$\int_0^1 dx\, u_m^*(x) u_n(x) = \int_0^1 dx \cdot (\sin m\pi x)(\sin n\pi x) = \tfrac{1}{2}\delta_{mn} \qquad (15)$$

Note that when $m = n$, $\int_0^1 dx \cdots \to \int_0^1 dx\, \sin^2 n\pi x = \int_0^1 dx \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2n\pi x\right) = \tfrac{1}{2}$ (16)

It follows from (16) that the appropriately normalized eigenfunctions are $u_n(x) = \sqrt{2}\,\sin(n\pi x)$.

Using Eq. (8) the solution for the unknown function $u(x)$ is given by

$$u(x) = \sum_n \beta_n u_n = \sum_n \frac{\alpha_n}{\lambda_n} u_n = \sum_n \Big\{ \underbrace{\frac{1}{n^2\pi^2}}_{1/\lambda_n} \cdot \underbrace{\int_0^1 \sqrt{2}\,\sin(n\pi x') f(x')\,dx'}_{\alpha_n} \Big\} \otimes \underbrace{\sqrt{2}\,\sin(n\pi x)}_{u_n} \qquad (17)$$

Hence:

$$\boxed{u(x) = \frac{2}{\pi^2} \sum_n \frac{1}{n^2} \left(\int_0^1 dx' \sin(n\pi x') f(x') \right) \sin n\pi x} \qquad (18)$$

This solution clearly has the correct properties with respect to the boundary conditions, since it satisfies $u(1) = u(0) = 0$.

For our purposes it is convenient to rewrite $u(x)$ in the form

$$\boxed{u(x) = \int_0^1 dx' \underbrace{\left\{ \frac{2}{\pi^2} \sum_n \frac{1}{n^2} \cdot \sin(n\pi x') \sin(n\pi x) \right\}}_{G(x,x')} f(x')} \qquad (19)$$

GREEN'S FUNCTION

Note that $G(x,x') = G(x',x)$ which is a generic property of Green's functions. Eq. (19) shows the characteristic form of a Green's function solution:

$$L\,u(x) = f(x) \;\Rightarrow\; u(x) = \int_0^1 dx' \, \underbrace{G(x,x')}\, f(x') \qquad (20)$$

↳ depends only on L & b.c. but not on $f(x)$

Intuitively we can understand the form of the Green's function solution as follows. Start with:

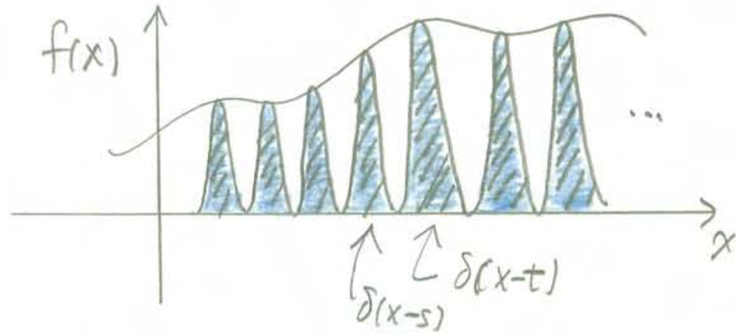
$$L u = f \;\Rightarrow\; \underbrace{L^{-1}(L u)}_{u} = L^{-1} f \;\Rightarrow\; \boxed{u = L^{-1} f} \qquad (21)$$

Comparing (20) & (21) we see that $L^{-1} \sim \int dx'\, G(x,x')$ (22)

which makes sense: <u>the inverse of the differential operator L is an integral operator.</u>

GENERAL APPROACH TO FINDING A GREEN'S FUNCTION

Intuitive Idea: We wish to solve the inhomogeneous equation $Lu(x) = f(x)$ for an arbitrary $f(x)$. The basic approach amounts to recognizing that an arbitrary function $f(x)$ can be viewed as an appropriate superposition of Dirac δ-functions:



Hence if we can solve the inhomogeneous equation for a δ-function source we can also find the solution for any other source using superposition. To see this formally we solve for

$$\boxed{\text{in the variable } x} \longrightarrow \boxed{L_x\, g(x,t) = \delta(x-t)} \quad ; \quad t = \text{fixed} \qquad (1)$$

Once $g(x,t)$ is known for this case we can write:

$$L_x u(x) = f(x) \Rightarrow \boxed{u(x) = \int dt\, g(x,t)\, f(t)} \qquad (2)$$

Check: From (2): $L_x u(x) = L_x \int dt\, g(x,t) f(t) = \int dt \underbrace{\left[L_x g(x,t)\right]}_{\delta(x-t)} f(t)$ (3)

$$= \int dt\, \delta(x-t) f(t) = f(x) \checkmark$$

This verifies that the solution in (2) is correct, where $g(x,t)$ is the solution to (1). The function $g(x,t)$ which solves an inhomogeneous equation as in (2) is called the GREEN'S FUNCTION for L_x (subject to appropriate boundary conditions.) Recalling from 167(2) that

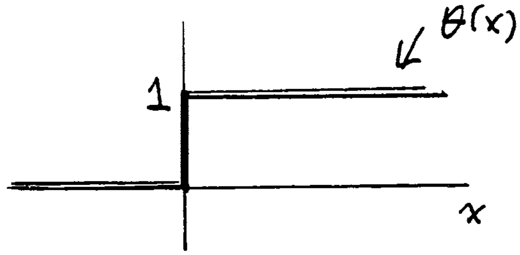
$$u = L^{-1} f = \int dt\, g(x,t) f(t) \Rightarrow \boxed{L^{-1} \sim \int dt\, g(x,t)} \qquad (4)$$

Note that Green's functions are not a fixed set of known functions like Legendre functions; rather each L_x and its associated boundary conditions lead to a different Green's function.

This technique is useful because the Green's functions for common differential operators with typical boundary conditions can be found once and for all; for example $\nabla^2 \phi(\vec{r}) = \delta^3(\vec{r}-\vec{r}\,')$, with solutions vanishing at ∞. Once we find this Green's function we can in principle solve any equation of the form $\nabla^2 \phi(\vec{r}) = \rho(\vec{r})$.

To proceed we recall from last semester that the Dirac δ-function had the following properties

- $$\int_{-\infty}^{\infty} dx\; \delta(x-a)\, f(x) = f(a) \qquad (5a)$$
- $$(x-a)\,\delta(x-a) = 0 \qquad (5b)$$ ← in an integral...
- $$\int_{-\infty}^{\infty} dx\; f(x)\, \delta^{n}(x-a) = (-1)^n \left.\frac{d^n f}{dx^n}\right|_{x=a} \qquad (5c)$$
- $$\frac{d\theta(x)}{dx} = \delta(x) \qquad (5d)$$



- $$\theta(x) = 1 - \theta(-x) \qquad (5e)$$

We next illustrate this approach to Green's functions by returning to the problem on p.166, which we previously solved by the series method. We want to solve $L\,u(x) = f(x)$ where $L = d^2/dx^2$. From the previous discussion we then wish to solve

$$L_x g(x,t) = \frac{d^2}{dx^2}\, g(x,t) = \delta(x-t) \tag{6}$$

Integrating (6) $\Rightarrow$

$$\frac{d\,g(x,t)}{dx} = \underbrace{\int dx\ \delta(x-t)}_{\theta(x-t)} + \alpha(t) \leftarrow \text{x-independent integration constant} \tag{7}$$

Integrating a second time gives:

$$g(x,t) = \underbrace{\int dx\ \theta(x-t)}_{(x-t)\,\theta(x-t)} + x\,\alpha(t) + \beta(t) \tag{8}$$

Hence altogether:

$$\boxed{g(x,t) = (x-t)\,\theta(x-t) + x\,\alpha(t) + \beta(t)} \tag{9}$$

Check: $\frac{d}{dx} g(x,t) = \left\{ (x-t)\underbrace{\frac{d}{dx}\theta(x-t)}_{\delta(x-t)} + \theta(x-t) \right\} + \alpha(t) = \underbrace{(x-t)\,\delta(x-t)}_{\to 0} +$ [redacted] $\theta(x-t) + \alpha(t)$ ✓ (10)

$$\frac{d^2 g(x,t)}{dx^2} = \delta(x-t) \ ✓ \tag{11}$$

At this point $g(x,t)$ in (9) is determined up to <u>2 integration constants $\alpha(t)$, $\beta(t)$</u> which must now be fixed using the boundary conditions. Since the boundary conditions apply to the solutions $u(x)$ [and not directly to $g(x,t)$] we write

$$u(x) = \int_{-\infty}^{\infty} dt\ g(x,t)\,f(t) = \int_{-\infty}^{\infty} dt \left\{ (x-t)\,\theta(x-t) + x\,\alpha(t) + \beta(t) \right\} f(t)$$

$$= \int_{-\infty}^{\infty} dt\,(x-t)\,\theta(x-t)\,f(t) + x\int_{-\infty}^{\infty} dt\ \alpha(t)\,f(t) + \int_{-\infty}^{\infty} dt\ \beta(t)\,f(t) \tag{12}$$

Note that since $\theta(x-t)=1$ when $x>t$ we can write:

$$\int_{-\infty}^{\infty} dt\,(x-t)\,\theta(x-t)\,f(t) = \int_{-\infty}^{x} dt\,(x-t)\,f(t) = x\int_{-\infty}^{x} dt\cdot f(t) - \int_{-\infty}^{x} dt\cdot t\,f(t) \qquad (13)$$

Hence at this stage:

$$u(x) = x\int_{-\infty}^{x} dt\cdot f(t) - \int_{-\infty}^{x} dt\cdot t\,f(t) + x\int_{-\infty}^{\infty} dt\,\alpha(t)\,f(t) + \int_{-\infty}^{\infty} dt\,\beta(t)\,f(t) \qquad (14)$$

We next impose the boundary condition $\underline{\underline{u(x=0)=0}} \Rightarrow$

$$0 = -\int_{-\infty}^{0} dt\; t\,f(t) + \int_{-\infty}^{\infty} dt\,\beta(t)\,f(t) \qquad (15)$$

By inspection we see that a solution to (15) is

$$\boxed{\beta(t) = t\,\theta(-t)} \qquad (16)$$

$\underline{\text{Check:}}$

$$-\int_{-\infty}^{0} dt\; t\,f(t) + \int_{-\infty}^{\infty} dt\,[t\,\theta(-t)]\,f(t) = -\int_{-\infty}^{0} dt\; t\,f(t) + \int_{-\infty}^{0} dt\,[t\cdot 1]\,f(t) = 0. \checkmark \qquad (17)$$

We next impose the boundary condition $\underline{\underline{u(x=1)=0}}$. From (14) & (16) this gives

$$0 = 1\cdot\int_{-\infty}^{1} dt\,f(t) - \int_{-\infty}^{1} dt\cdot t\,f(t) + 1\cdot\underline{\underline{\int_{-\infty}^{\infty} dt\,\alpha(t)\,f(t)}} + \underbrace{\int_{-\infty}^{\infty} dt\,[t\,\theta(-t)]\,f(t)}_{\int_{-\infty}^{0} dt\; t\,f(t) \Leftarrow (17)} \qquad (18)$$

$\underline{\text{Side Comment:}}$ We see from (18) that in order to determine $\alpha(t)$ all we need is the integral obtained from (15) or (17), not necessarily $\beta(t)$ itself.

Combining the terms indicated by $\sim\sim$ gives:

$$-\int_{-\infty}^{1}\cdots + \int_{-\infty}^{0}\cdots = -\int_{0}^{1}\cdots \qquad (19)$$

So (18) & (19) $\Rightarrow$

$$0 = \int_{-\infty}^{1} dt\,f(t) - \int_{0}^{1} dt\; t\,f(t) + \underline{\underline{\int_{-\infty}^{\infty} dt\,\alpha(t)\,f(t)}} \qquad (20)$$

As before, (20) evaluates $\int dt\,\alpha(t)\,f(t)$ which is all we need in (18).

At this stage we can collect the previous results together:

$$u(x) = \underbrace{x\int_{-\infty}^{x} dt\, f(t) - \int_{-\infty}^{x} dt\cdot t f(t)}_{\int_{-\infty}^{x} dt\,(x-t)\, f(t)} + x\left\{\underbrace{\int_{0}^{1} dt\, t f(t) - \int_{-\infty}^{1} dt\, f(t)}_{\int dt\, \alpha(t) \ldots}\right\} + \underbrace{\int_{-\infty}^{0} dt\, t f(t)}_{\int \beta(t)\, dt \ldots} \qquad (21)$$

$$\hookrightarrow \int_{-\infty}^{x} = \int_{-\infty}^{0} + \int_{0}^{x}$$

$$\therefore\ u(x) = \int_{-\infty}^{0} dt\,(\cancel{x}-\cancel{t}) f(t) + \int_{0}^{x} dt\,(x-t) f(t) + x\int_{0}^{1} dt\cdot t f(t) \cancel{- x\int_{-\infty}^{0} dt\, f(t)} - x\int_{0}^{1} dt\, f(t) + \int_{-\infty}^{0} dt\, \cancel{t} f(t) \qquad (22)$$

$$\therefore\ u(x) = \int_{0}^{x} dt\,(x-t) f(t) + x\int_{0}^{1} dt\,(t-1) f(t)$$

$$\therefore\ \boxed{u(x) = \int_{0}^{1} dt\,\big[(x-t)\,\theta(x-t) - x(1-t)\big] f(t) \equiv \int_{0}^{1} dt\, g(x,t)\, f(t)} \qquad (23)$$

This is the Green's function $g(x,t)$ for $L = d^2/dx^2$ subject to the boundary conditions $u(1) = u(0) = 0$. Note that since the only dependence of $u(x)$ on x on the r.h.s. of (23) is in $g(x,t)$, and hence it must be that $g(x,t)$ satisfies the boundary conditions as well:

$$g(x=0,t) = -t\,\theta(-t) = 0 \quad (\text{since } -t<0 \text{ in } [0,1] \Rightarrow \theta(-t)=0) \ \checkmark \qquad (24)$$

$$g(1,t) = \{(1-t)\,\theta(1-t) - (1-t)\} = (1-t)\,[\theta(1-t) - 1] = -(1-t)\,[1-\theta(1-t)]$$

$$169(5e) \Rightarrow \quad = -(1-t)\,\theta(t-1) \qquad (25)$$

But in the range $1 \geq t \geq 0$, $\theta(t-1) \equiv 0$. ✓

$\therefore\ g(1,t) = 0$.

Check on $g(x,t)$: Suppose that we wish to solve $d^2u(x)/dx^2 = c$ where c = constant; we know immediately that the solution can be found by writing

$$u(x) = \frac{1}{2}cx^2 + \alpha x + \beta \quad (\text{this} \Rightarrow d^2u(x)/dx^2 = c) \tag{26}$$

$$u(0) = 0 \Rightarrow \beta = 0 \;;\; u(1) = 0 \Rightarrow 0 = \frac{1}{2}c + \alpha \Rightarrow \alpha = -\frac{1}{2}c \tag{27}$$

$$\boxed{\therefore\; u(x) = \frac{1}{2}cx^2 - \frac{1}{2}cx = \frac{1}{2}cx(x-1)} \tag{28}$$

By comparison, the Green's function solution in (23) gives:

$$u(x) = \int_0^1 dt\,\left[(x-t)\,\theta(x-t) - x(1-t)\right]\cdot \underset{\substack{\uparrow \\ f(x)=c}}{c} = c\left\{\int_0^x dt\,(x-t) - x\left[t - \frac{1}{2}t^2\right]_0^1\right\} \tag{29}$$

$$= c\left[xt - \frac{1}{2}t^2\right]_0^x - cx\left(1 - \frac{1}{2}\right) = c\left[x^2 - \frac{1}{2}x^2\right] - \frac{1}{2}cx = \frac{1}{2}cx(x-1) \tag{30}$$

GREEN'S FUNCTION FOR THE POISSON EQUATION

We want to solve the Poisson equation

$$-\nabla^2 u(\vec{x}) = f(\vec{x}) \tag{1}$$

As before we look at the solution of the equation

$$-\nabla^2 G(\vec{x}, \vec{x}') = \delta^3(\vec{x} - \vec{x}') \tag{2}$$

The solution to (1) will then be given by

$$\boxed{u(\vec{x}) = \int d^3x' \, G(\vec{x}, \vec{x}') \, f(x')} \tag{3}$$

"superposition of δ-functions"

Check: $$-\nabla^2 u(\vec{x}) = -\nabla_x^2 u(\vec{x}) = \int d^3x' \Big[\underbrace{-\nabla_x^2 G(\vec{x}, \vec{x}')}_{\delta^3(\vec{x}-\vec{x}')}\Big] f(x') = f(\vec{x}) \checkmark \tag{4}$$

First we solve the simple case of a charge at the origin so that $\vec{x} - \vec{x}' \to \vec{x}$. Then (2) assumes the form $-\nabla^2 G(\vec{x}, 0) = \delta^3(\vec{x})$. Since ∇^2 is spherically symmetric it is convenient to work in polar coordinates such that

$$\nabla^2 G = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial G}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 G}{\partial \phi^2} \tag{5}$$

Claim: In polar coordinates $\delta^3(\vec{x})$ can be replaced by

$$\delta^3(x) \longrightarrow \frac{\delta(r)}{4\pi r^2} \tag{6}$$

Check: $$I = \int d^3x \, h(\vec{x}) \, \delta^3(\vec{x}) = h(0) \checkmark = \int_0^\infty dr \cdot r^2 \underbrace{\int_0^\pi d\theta \cdot \sin\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \, h(r, \theta, \phi) \cdot \frac{\delta(r)}{4\pi r^2}$$

$$= 4\pi \int_0^\infty dr \cdot r^2 \frac{\delta(r)}{4\pi r^2} \underbrace{h(0, \theta, \phi)}_{\equiv h(0)} = h(0) \checkmark \tag{7}$$

Note that once $r = 0$ is fixed by $\delta(r)$, θ and ϕ can have any value at the origin, so we can set $\theta = \phi = 0$.

From the preceding discussion the Green's function equation that we want to solve is

$$\nabla^2 G(r) = \underbrace{-\frac{1}{4\pi r^2}\,\delta(r)}_{\text{independent of }\theta,\phi} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 G}{\partial\phi^2}}_{\text{depends on }\theta,\phi} \qquad (8)$$

It follows that the only way that (8) can hold is:

$$\frac{\partial G}{\partial\theta} = \frac{\partial G}{\partial\phi} = 0 \qquad (9)$$

Of course this makes sense, since we expect on symmetry grounds that for a point charge at the origin G should be independent of θ, ϕ. Then (8), (9) $\Rightarrow$

$$\nabla^2 G(r) = -\frac{1}{4\pi \cancel{r^2}}\,\delta(r) = \frac{1}{\cancel{r^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) \Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = -\frac{1}{4\pi}\,\delta(r) \qquad (10)$$

Now $G(r)$ is expected to be a well-behaved function, except possibly at $r=0$. Hence away from $r=0$ Eq. (10) $\Rightarrow$ that $G(r)$ must be a solution of the equation

$$\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = 0 \Rightarrow r^2\frac{dG}{dr} = \text{constant} \equiv -a \Rightarrow \frac{dG}{dr} = -\frac{a}{r^2} \qquad (11)$$

This gives: $\boxed{G(r) = \frac{a}{r} + b}$ (12)

<u>Boundary Conditions</u>: Note that we have to fix 2 constants, since we started from a second-order differential equation: To fix b we note that $G(r)$ is the solution to Poisson's equation for a charge at the origin, so that the potential should vanish at $r\to\infty \Rightarrow \boxed{b=0}$. To fix a we note that $G(r)$ is a solution of

$$\nabla^2 G(r) = -\delta^3(\vec{r}) = -\frac{1}{4\pi r^2}\,\delta(r) \qquad (13)$$

From (12):

$$\nabla^2 G(r) = a\nabla^2\left(\frac{1}{r}\right) = a\left[-4\pi\delta^3(\vec{r})\right] = -4\pi a\cdot\frac{1}{4\pi r^2}\,\delta(r) \qquad (14)$$

We see that (13) & (14) are compatible if we choose $\boxed{a = 1/4\pi}$ $\boxed{\text{III-176,177}}$

Hence altogether:

$$\boxed{G(r) = \frac{1}{4\pi r} \Rightarrow G(\vec{x}-\vec{x}') = \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}} \quad (15)$$

Combining (15) with Eq. (3) above we see that $\nabla^2 u(\vec{x}) = f(x)$ is solved by

$$\boxed{u(\vec{x}) = \int d^3x' \, f(x') \frac{1}{4\pi\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}}} \quad (16)$$

- This is the same solution that we found last semester for Poisson's equation with the identification $u(\vec{x}) \to \phi(\vec{x})$ and $f(\vec{x}') \to \rho(\vec{x}')$.

- If we were to replace the integral by a sum, then it would be intuitively clear that we are adding the individual contributions of point charges located at $\vec{x}'$, weighted by the amplitude $f(\vec{x}')$. Hence the Green's function $1/4\pi r$ is just the Coulomb potential (up to a factor of the electric charge e).

GENERAL SOLUTION OF POISSON'S EQUATION

We outline here the general solution of Poisson's equation for a source charge located at an arbitrary point $\vec{x}'$. The detailed steps are to be filled in for homework.

We want to solve $\nabla^2 G(\vec{x}, \vec{x}') = -\delta^3(\vec{x} - \vec{x}')$ (1)

In spherical coordinates this gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial G}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 G}{\partial \phi^2} = -\frac{\delta(r-r')}{r^2}\delta(\cos\theta - \cos\theta')\delta(\phi-\phi') \quad (2)$$

Expressing $\delta^3(\vec{x} - \vec{x}')$ is conveniently done in terms of $\delta(\cos\theta - \cos\theta')$ rather than in terms of $\delta(\theta - \theta')$ because this allows us to use directly the completeness relation for the spherical harmonics $Y_{\ell m}$:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi')\, Y_{\ell m}(\theta, \phi) = \delta(\cos\theta - \cos\theta')\,\delta(\phi - \phi') \quad (3)$$

Using (3) the r.h.s. of (2) can be then expressed as

$$\boxed{\delta^3(\vec{x} - \vec{x}') = -\frac{1}{r^2}\delta(r-r') \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi')\, Y_{\ell m}(\theta, \phi)} \quad (4)$$

We next assume* that $G(\vec{x}, \vec{x}')$ can be expanded in the form

$$G(\vec{x}, \vec{x}') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_\ell(\vec{r}, \vec{r}')\, Y^*_{\ell m}(\theta', \phi')\, Y_{\ell m}(\theta, \phi) \quad (5)$$

*As usual the justification for this assumption is that we will show in the end that IT WORKS!

Inserting Eq. (5) into Eq. (2) we find after some algebra (do for homework!!) that $g_\ell(r,r')$ satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_\ell}{dr}\right)-\frac{\ell(\ell+1)}{r^2}g_\ell=-\frac{1}{r^2}\delta(r-r') \qquad (6)$$

From Eq. (5) we see that $G(\vec{x},\vec{x}')$ is determined once we solve for $g_\ell=g_\ell(r,r')$.

To find the solution for g_ℓ, first solve the homogeneous equation in (6):

When $r\neq r'$, $\delta(r-r')=0$, and hence whatever solution we find by doing this should be the correct solution when $r\neq r'$! The homogeneous equation can then be written as:

$$\frac{d^2}{dr^2}(rg_\ell)-\frac{\ell(\ell+1)}{r^2}(rg_\ell)=0 \quad ; \quad rg_\ell\equiv u_\ell \qquad (7)$$

$$\therefore \quad \frac{d^2u_\ell}{dr^2}-\frac{\ell(\ell+1)}{r^2}u_\ell=0 \qquad (8)$$

This equation has the general solution: $u_\ell=Ar^{\ell+1}+Br^{-\ell}$ (9)

Hence ($u_\ell=rg_\ell$)

$$g_\ell=\begin{cases}Ar^\ell+Br^{-\ell-1} & r<r' \\ A'r^\ell+B'r^{-\ell-1} & r>r'\end{cases} \qquad (10)$$

The fact that we allow for different solutions for $r<r'$ and $r>r'$ reflects the δ-function singularity at $r=r'$! The constants A, B, A', B' can be determined by the following boundary conditions:

(a) Since we are solving the homogeneous equation (no charges present) we want the solution to be well-behaved at $r=0$. $\Rightarrow \boxed{B=0}$ (11a)

(b) The solution to g_ℓ should vanish at $r\to\infty$. $\Rightarrow \boxed{A'=0}$ (11b)

Hence at this stage we have:

$$g_\ell(r,r') = \begin{cases} A r^\ell & r<r' \\ B' r^{-\ell-1} & r>r' \end{cases} \qquad (12)$$

The constants A and B' are now determined by matching the solutions in (12) when $r=r'$. We expect that $g_\ell(r,r')$ should be continuous at $r=r'$, but not $dg_\ell(r,r')/dr$. Continuity of g_ℓ at $r=r'$ then gives:

$$A(r')^\ell = B'(r')^{-\ell-1} \Rightarrow \boxed{A = B'(r')^{-2\ell-1}} \qquad (13)$$

Since r' is a constant this relates the unknown constants A, B' in terms of each other.

Finally we must deal with the <u>discontinuity at $r=r'$ due to $\delta(r-r')$.</u>

Consider the general case : $\dfrac{d}{dx}\left[p\dfrac{dG}{dx}\right] - s(x)\,G = \delta(x-\xi)$ (14)

From our Eq. (6) above $p=r^2$ and $s(x)=\ell(\ell+1)$. (15)

Next integrate both sides of Eq. (14) across $x=\xi$:

$$\int_{\xi-\epsilon}^{\xi+\epsilon} dx\left\{\frac{d}{dx}\left[p\frac{dG}{dx}\right]\right\} - \underbrace{\int_{\xi-\epsilon}^{\xi+\epsilon} dx\, s(x)\,G}_{0} = \int_{\xi-\epsilon}^{\xi+\epsilon} dx\,\delta(x-\xi) = 1 \qquad (16)$$

$= 0$, since $s(x)$ and G are continuous at $x=\xi$

$$\left. p\frac{dG}{dx}\right|_{\xi-\epsilon}^{\xi+\epsilon} \overset{*}{=} p\left[\frac{dG(\xi+\epsilon,\xi)}{dx} - \frac{dG(\xi-\epsilon,\xi)}{dx}\right] = 1 \qquad (17)$$

* Note that $G(r,r') \longrightarrow G(\xi\pm\epsilon,\xi)$, since $r'=\xi$ is fixed.

Hence (17) $\Rightarrow$

$$\boxed{\frac{dG(\xi+\epsilon,\xi)}{dx} - \frac{dG(\xi-\epsilon,\xi)}{dx} = \frac{1}{p(\xi)}} \qquad (18)$$

In the present case we have from Eq. (12):

$$\frac{dg_\ell}{dr} = \begin{cases} A\ell\, r^{\ell-1} & r<r' \\ -B'(\ell+1)\, r^{-\ell-2} & r>r' \end{cases} \qquad (19)$$

Noting that $p = r^2$ in our case, we find at r':

$$A\ell (r')^{\ell-1} + B'(\ell+1)(r')^{-\ell-2} = \frac{1}{(r')^2} \Rightarrow \boxed{A\ell (r')^{\ell+1} + B'(\ell+1)(r')^{-\ell} = 1} \qquad (20)$$

Eqs. (20) & (13) can now be solved for A & B':

$$\boxed{A = \frac{(r')^{-\ell-1}}{2\ell+1} \qquad B' = \frac{(r')^{\ell}}{2\ell+1}} \qquad (21)$$

Hence

$$g_\ell(r,r') = \begin{cases} \dfrac{1}{2\ell+1}\,\dfrac{r^\ell}{(r')^{\ell+1}} & r<r' \\[2ex] \dfrac{1}{2\ell+1}\,\dfrac{(r')^\ell}{r^{\ell+1}} & r>r' \end{cases} \equiv \frac{1}{2\ell+1}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right) \qquad (22)$$

Finally, then:

$$\boxed{G(\vec{x},\vec{x}\,') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right) Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)} \qquad (23)$$

GREEN'S FUNCTIONS FOR DIFFUSION-TYPE EQUATIONS

This shows how we can generate the Green's function for an operator knowing the eigenfunctions and eigenvalues of that operator.

Consider as an example the <u>heat equation</u>

$$\nabla^2 T(\vec{x},t) = \frac{c}{k}\frac{\partial T(\vec{x},t)}{\partial t} \Rightarrow \frac{\partial T}{\partial t} = \frac{k}{c}\nabla^2 T \qquad (1)$$

T= absolute temperature, c= specific heat/volume, k= heat conductivity

There are several similar equations which can be cast into the form

$$H\psi(\vec{x},\tau) = -\frac{\partial\psi(\vec{x},\tau)}{\partial\tau} \qquad (2)$$

NAME	H	$\psi(\vec{x},\tau)$	τ
1) heat	$-\nabla^2$	T = temperature	$(k/c)\,t$
2) diffusion	$-\nabla^2$	ρ = density	$a^2 t$; a = diffusion coefficient
3) Schrödinger	$-\nabla^2$	ψ = probability amplitude	$(i\hbar^2/2m)\,t$

Note that the diffusion & heat equations are not invariant under <u>time-reversal</u>. However, the Schrödinger equation is, because when we let $t\to -t$ we also complex conjugate the operators & wavefunctions.

Since H is Hermitian its eigenfunctions form a complete orthonormal (CON) set:

$$H\phi_m = \lambda_m \phi_m \qquad m = 0, 1, 2, \ldots \qquad (3)$$

To specify the ϕ_m appropriate boundary conditions must be chosen for $\psi(\vec{x},\tau)$, as we discuss below.

We next assume that we can find a solution of (2) having the form

$$\psi(\vec{x},\tau) = \sum_m A_m(\tau)\,\phi_m(\vec{x}) \tag{4}$$

Substituting this into the differential equation (2) we find

$$H\psi(\vec{x},\tau) = \sum_m A_m(\tau)\,\underbrace{H\phi_m(\vec{x})}_{\lambda_m \phi_m(x)} = \sum_m A_m(\tau)\,\lambda_m \phi_m(\vec{x}) = -\sum_m \frac{\partial A_m(\tau)}{\partial \tau}\phi_m(\vec{x}) \tag{5}$$

Hence: $$\sum_m \left\{ A_m(\tau)\lambda_m + \frac{\partial A_m(\tau)}{\partial \tau} \right\}\phi_m = 0 \tag{6}$$

Since the $\phi_m(\vec{x})$, being a basis, are linearly independent, (6) $\Rightarrow$

$$A_m(\tau)\,\lambda_m + \frac{\partial A_m(\tau)}{\partial \tau} = 0 \Rightarrow \boxed{A_m(\tau) = A_m(0)\,e^{-\lambda_m \tau}} \tag{7}$$

Combining (4) & (7) $\Rightarrow$ $$\psi(\vec{x},\tau) = \sum_m A_m(0)\,e^{-\lambda_m \tau}\phi_m(\vec{x}) \tag{8}$$

The constants $A_m(0)$ are determined by the boundary condition which fixed $\psi(\vec{x},0)$:

$$\psi(\vec{x},0) = \sum_m A_m(0)\,\phi_m(x) \tag{9}$$

This can be inverted to solve for $A_m(0)$:

$$\int d^3x\, \phi_n^*(\vec{x})\,\psi(\vec{x},0) = \sum_m A_m(0) \int d^3x\, \underbrace{\phi_n^*(\vec{x})\,\phi_m(x)}_{\delta_{nm}} = A_n(0) \tag{10}$$

So: $$\boxed{A_n(0) = \int d^3x\, \phi_n^*(\vec{x})\,\psi(\vec{x},0)} \tag{11}$$

Combining (9) & (11): $$\psi(\vec{x},\tau) = \sum_m \underbrace{\left[\int d^3x'\, \phi_m^*(\vec{x}')\,\psi(\vec{x}',0)\right]}_{A_m(0)} \phi_m(\vec{x})\,e^{-\lambda_m \tau} \tag{11}$$

Finally:

$$\boxed{\psi(\vec{x},\tau) = \int d^3x' \underbrace{\left[\sum_m \phi_m^*(\vec{x}')\,\phi_m(\vec{x})\,e^{-\lambda_m \tau}\right]}_{G(\vec{x},\vec{x}',\tau)} \psi(\vec{x}',0)} \tag{12}$$

We refer to the expression in [...] as the Green's function because (as in the previous cases) it is the response of the system to a δ-function input. Specifically, suppose in Eq. (12) that

$$\text{input} = \psi(\vec{x}\,',0) = \delta^3(\vec{x}\,'-\vec{x}\,'') \tag{13}$$

Then $$\psi(\vec{x},\tau) = \int d^3x'\, G(\vec{x},\vec{x}\,',\tau)\,\psi(\vec{x}\,',0) = \int d^3x'\, G(\vec{x},\vec{x}\,',\tau)\,\delta^3(\vec{x}\,'-\vec{x}\,'') = G(\vec{x},\vec{x}\,'',\tau) \tag{14}$$

Hence $$\boxed{G(\vec{x},\vec{x}\,'',0) = \psi(\vec{x},0) = \delta^3(\vec{x}-\vec{x}\,'')} \tag{15}$$

This establishes that $G(\vec{x},\vec{x}\,'',0)$ is infact the solution to a δ-function input, and for this reason deserves to be called the Green's function.

We also see from (12) that the Green's function propagates the wavefunction from $\tau=0$ to τ, and for this reason it is referred to as the propagator in such applications. The Green's function $G(\vec{x},\vec{x}\,',\tau)$ obeys the following "group property":

$$\boxed{\int d^3x'\, G(\vec{x},\vec{x}\,',\tau_1)\, G(\vec{x}\,',\vec{x}\,'',\tau_2) = G(\vec{x},\vec{x}\,'',\tau_1+\tau_2)} \tag{16}$$

✓

Proof: The l.h.s. of (16) is given by (dropping vector signs for simplicity)

$$\text{l.h.s.} = \int d^3x' \left[\sum_m \phi_m^*(x')\phi_m(x)\, e^{-\lambda_m\tau_1}\right]\left[\sum_n \phi_n^*(x'')\phi_n(x')\, e^{-\lambda_n\tau_2}\right] \tag{17}$$

$$= \sum_{m,n} \phi_n^*(x'')\phi_m(x)\, e^{-\lambda_m\tau_1-\lambda_n\tau_2} \underbrace{\int d^3x'\, \phi_m^*(x')\phi_n(x')}_{\delta_{mn}} = \sum_m \phi_m^*(x'')\phi_m(x)\, e^{-\lambda_m(\tau_1+\tau_2)} \tag{18}$$

$$= G(x,x'',\tau_1+\tau_2) \checkmark$$

Interpretation: Starting from $\psi(\vec{x},0)$, $G(\vec{x},\vec{x}\,',\tau_1)$ propagates this solution forward from $\tau=0$ to $\tau=\tau_1$:

$$\psi(\vec{x},\tau_1) = \int d^3x'\, G(\vec{x},\vec{x}\,',\tau_1)\,\psi(\vec{x}\,',0) \tag{19}$$

known input

Once we know $\psi(\vec{x}, \tau_1)$ we use $G(\vec{x}, \vec{x}'', \tau_2)$ to propagate this solution forward in time to $\tau_1 + \tau_2$:

$$\psi(\vec{x}, \tau_1+\tau_2) = \int d^3x'' \, G(\vec{x}, \vec{x}'', \tau_2) \, \psi(\vec{x}'', \tau_1) \tag{20}$$

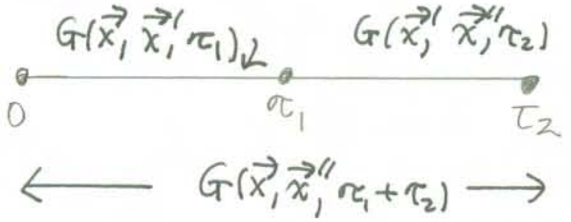
known input from (19) (let $\vec{x} \to \vec{x}''$)

$$\text{Hence: } \psi(\vec{x}, \tau_1+\tau_2) = \int d^3x'' \, G(\vec{x}, \vec{x}'', \tau_2) \left[\int d^3x' \, G(\vec{x}'', \vec{x}', \tau_1) \, \psi(\vec{x}', 0) \right] \tag{21}$$

$$= \int d^3x' \underbrace{\left\{ \int d^3x'' \, G(\vec{x}, \vec{x}'', \tau_2) \, G(\vec{x}'', \vec{x}', \tau_1) \right\}}_{\longleftarrow \; G(\vec{x}, \vec{x}', \tau_1+\tau_2) \; \longrightarrow} \psi(\vec{x}', 0) \tag{22}$$

Hence finally
$$\boxed{\psi(\vec{x}, \tau_1+\tau_2) = \int d^3x' \, G(\vec{x}, \vec{x}', \tau_1+\tau_2) \, \psi(\vec{x}', 0)} \tag{23}$$

Pictorially:



Detailed Form of the Green's Function

We want to find the detailed functional form of the Green's function for the diffusion equation. We begin by solving the eigenvalue problem subject to the appropriate boundary conditions. We quantize in a 1-dim box of side L such that $\psi(L/2) = \psi(-L/2)$. $H = -\partial^2/\partial x^2 \Rightarrow \psi(x,\tau)$ solves

$$\partial^2 \psi(x,\tau)/\partial x^2 = \partial \psi/\partial \tau \qquad \tau = \text{REAL} \tag{24}$$

We have previously shown [p. 180, 181] that if we assume that

$$\psi(x,\tau) = \sum_m A_m(\tau) \, \phi_m(x) \Rightarrow A_m(\tau) = A_m(0) \, e^{-\lambda_m \tau} \tag{25}$$

Combining (24) & (25) $\Rightarrow$
$$\frac{\partial^2 \psi}{\partial x^2} = \sum_m A_m(0) \cancel{e^{-\lambda_m \tau}} \frac{\partial^2 \phi_m}{\partial x^2} = \sum_m A_m(0) \underbrace{\left(-\lambda_m \cancel{e^{-\lambda_m \tau}}\right)}_{\partial \psi/\partial \tau} \phi_m \tag{26}$$

Hence ϕ_m is a solution of

$$\sum_m \left(\frac{\partial^2\phi_m}{\partial x^2} + \lambda_m \phi_m\right) = 0 \tag{27}$$

Since the ϕ_m are linearly independent this must hold separately for each m. $\Rightarrow$

$$\frac{\partial^2\phi_m}{\partial x^2} + \lambda_m\phi_m = 0 \Rightarrow \phi_m(x) = \frac{1}{\sqrt{L}}\, e^{i\sqrt{\lambda_m}\,x} \tag{28}$$

To satisfy the boundary condition $\psi(-L/2) = \psi(L/2)$ we write

$$\phi_m(L/2) = (1/\sqrt{L})\, e^{i\sqrt{\lambda_m}(L/2)} = (1/\sqrt{L})\, e^{-i\sqrt{\lambda_m}(L/2)} \Rightarrow \boxed{\sqrt{\lambda_m} = \frac{2\pi m}{L}\,;\ m = 0, \pm 1, \ldots} \tag{29}$$

$$\therefore\ \lambda_m = (2\pi m/L)^2 \equiv k_m^2 \Rightarrow \boxed{A_m(\tau) = A_m(0)\, e^{-k_m^2\tau}} \tag{30}$$

Then:

$$\psi(x,\tau) = \sum_m A_m(\tau)\phi_m(x) = \sum_m A_m(0)\, e^{-k_m^2\tau}\, \frac{1}{\sqrt{L}}\, e^{ik_m x} \tag{31}$$

From Eq. (12) p. 180, 181 we see that the 1-dimensional Green's function is given by:

$$G(x,x',\tau) = \sum_m \phi_m^*(x')\phi_m(x)\, e^{-\lambda_m\tau} = \frac{1}{L}\sum_m e^{-ik_m x'}\, e^{ik_m x}\, e^{-k_m^2\tau} \tag{32}$$

Finally:

$$\boxed{G(x,x',\tau) = \frac{1}{L}\sum_{m=-\infty}^{\infty} e^{ik_m(x-x')}\, e^{-k_m^2\tau}} \tag{33}$$

CONVERTING SUMS TO INTEGRALS: GENERAL DISCUSSION

For some calculational purposes it may be convenient to start a problem by quantizing in a 1-dim box of side L, a 2-dim box of area L^2, or a 3-dim box of volume L^3. But we live in the real world, which corresponds to $L \to \infty$! In this case we approach the continuum limit, and we want to rewrite our formulas to reflect this.

Schematically, Eq. (33) has the form

$$\sum_{m=-\infty}^{\infty} \frac{1}{L} F(k_m) = \sum_{m=-\infty}^{\infty} \frac{\Delta m}{L} F(k_m); \quad \text{this follows by noting that } \Delta m = 1 \tag{34}$$

Since $k_m = \frac{2\pi m}{L} \Rightarrow \boxed{\Delta k = \frac{2\pi}{L} \Delta m}$ (again, $\Delta m = 1$) (35)

It follows that as $L \to \infty$ $\Delta k \to 0$ such that $L \Delta k = 2\pi =$ constant. In the limit $L \to \infty$ we can also write $k_m \to k =$ continuous variable. It then follows that

$$\lim_{L\to\infty} \sum_{m=-\infty}^{\infty} \frac{1}{L} F(k_m) = \lim_{L\to\infty} \sum_{m=-\infty}^{\infty} \frac{\Delta m}{L} F(k) = \lim_{L\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{\cancel{L}} \underbrace{\left(\frac{L\Delta k}{2\pi}\right)}_{\Delta m} F(k) \tag{36}$$

Hence $$\lim_{L\to\infty} \sum_{m=-\infty}^{\infty} \frac{1}{L} F(k_m) \to \frac{1}{2\pi} \sum_k \Delta k \, F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, F(k) \tag{37}$$

Schematically: $$\boxed{\lim_{L\to\infty} \frac{1}{L} \sum_{m=-\infty}^{\infty} \cdots = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \cdots} \tag{38}$$

Similarly in 3-dimensions: $$\boxed{\lim_{L\to\infty} \frac{1}{L^3 = V} \sum_{m_1, m_2, m_3 = -\infty}^{\infty} \cdots = \frac{1}{(2\pi)^3} \int d^3k \cdots} \tag{39}$$

Advantage of Box Quantization: Going to the continuum limit introduces factors of (2π) which replace factors of $1/L$. Although all (artificial!) factors of $(1/L)$ must cancel in the end, factors of 2π need not, since they also arise in "legitimate" ways, and end up in the final results. Hence using the "artificial" box normalization has the advantage that it allows a check on a computation, by ensuring that factors of $1/L$ cancel out in the end.

Returning to $G(x,x',\tau)$ in (33) we can now use (38) to write

$$G(x,x',\tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{-k^2\tau} \tag{40}$$

To evaluate the integral in (40) we complete the square in the exponential:

$$ik(x-x') - k^2\tau = -\tau\left[k - \frac{i(x-x')}{2\tau}\right]^2 - \frac{(x-x')^2}{4\tau} \tag{41}$$

Hence:

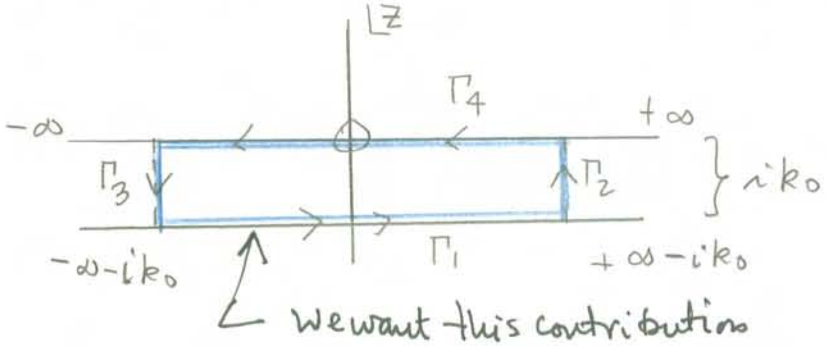
$$G(x,x',\tau) = e^{-(x-x')^2/4\tau}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{-\tau(k-ik_0)^2} \quad ; \; k_0 \equiv \frac{(x-x')}{2\tau} \tag{42}$$

To continue, let $z = k - ik_0$; $dz = dk$; $k=-\infty \Rightarrow z = -\infty - ik_0$
$k = +\infty \Rightarrow z = +\infty - ik_0$

Hence:

$$\boxed{G(x,x',\tau) = e^{-(x-x')^2/4\tau}\int_{-\infty-ik_0}^{\infty-ik_0}\frac{dz}{2\pi}\, e^{-\tau z^2}} \tag{43}$$

We can integrate the expression in (43) around the contour shown below:



$\Gamma_2, \Gamma_3 \to \infty$ eventually

We have: $\oint = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} = 0$ (no singularities inside contour) (44)

We now wish to argue that the contributions from Γ_2 and Γ_3 vanish separately. Consider Γ_3; for any finite value X we can write

$$\int_{\Gamma_3} = \int_{X-ik_0}^{X}\frac{dz'}{2\pi}\, e^{-\tau z'^2} = e^{-\tau X^2}\int_{X-ik_0}^{X} dy\, e^{\tau y^2}\, e^{i2\tau X y} \tag{45}$$

Note: $z' = x+iy \quad z'^2 = x^2 - y^2 + 2ixy \Rightarrow e^{-\tau z'^2} = e^{-\tau(x^2-y^2+2ixy)}$

Along Γ_3 η stays finite; hence since $x \to \infty$, the overall factor $e^{-\tau x^2}$ makes $\int_{\Gamma_3} \to 0$. The same holds true along Γ_2.

Hence $$\oint = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} \to \int_{\Gamma_1} + \int_{\Gamma_4} = 0 \Rightarrow \underbrace{\int_{\Gamma_1}}_{\text{we want}} = -\int_{\Gamma_4} \qquad (46)$$

So we can write

$$\int_{\Gamma_1} = -\int_{\Gamma_4} = -e^{-(x-x')^2/4\tau}\left(\frac{1}{2\pi}\right)\underbrace{\int_{\infty}^{-\infty} dx' e^{-\tau x'^2}}_{-\sqrt{\pi/\tau}} \qquad (47)$$

Hence altogether:

$$\boxed{G(x, x', \tau) = \frac{1}{\sqrt{4\pi\tau}}\, e^{-(x-x')^2/4\tau}} \qquad (48)$$

Note that $G(x, x', \tau) = G(x', x, \tau)$; G is symmetric in $x \leftrightarrow x'$

Physical Interpretation of the Solution:

Focus on the actual diffusion equation where $\tau = a^2 t$ (a^2=diffusion coeff.), and $\psi = \rho$ = density of material. For simplicity choose units so that $\tau = t$. Note to start with that $G(x, x', \tau)$ is normalized to unity so that

$$\int_{-\infty}^{\infty} dx'\, G(x, x', \tau) = \frac{1}{\sqrt{4\pi t}} \underbrace{\int_{-\infty}^{\infty} dx'\, e^{-(x-x')^2/4t}}_{\sqrt{4\pi t}} = 1 \qquad (49)$$

Note that the Green's function in (48) is a Gaussian whose width σ is related to τ by

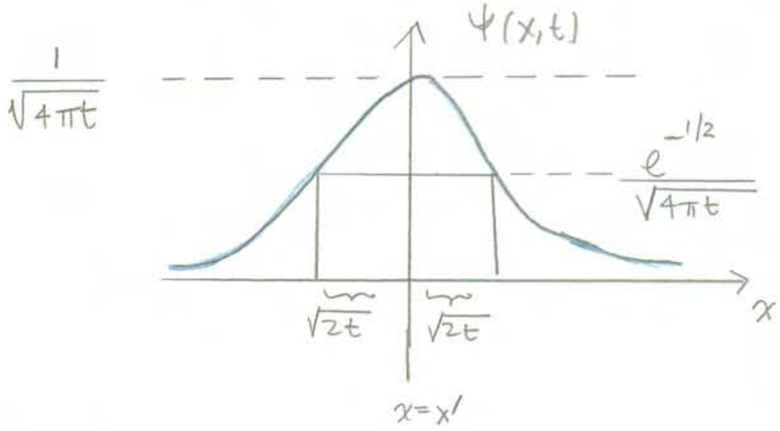
$$\sigma = \sqrt{2t} \qquad (50)$$

Hence the width of the Gaussian, which is proportional to $\sigma = \sqrt{2t}$ gets narrower as $t \to 0$.

However, since the normalization integral in (49) is independent of t it must be the case that

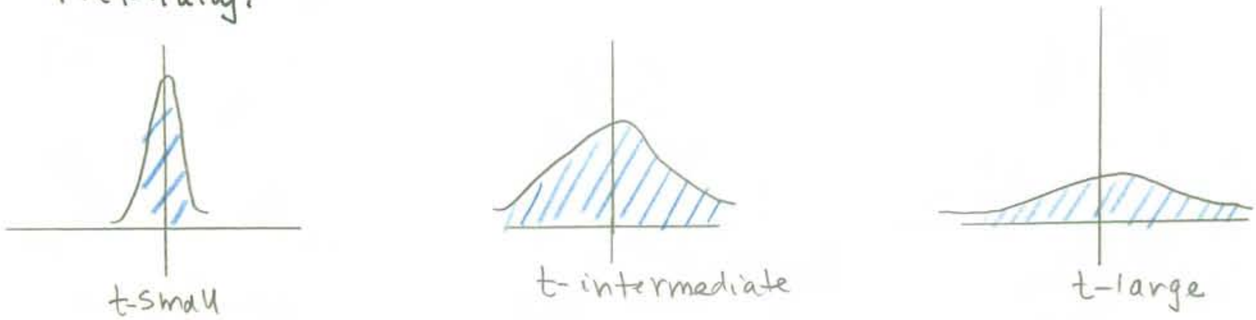
$$\boxed{\lim_{t\to 0} G(x,x',t) = G(x,x',0) = \delta(x-x')} \quad (51)$$

This agrees with Eq. (15), p. 181, 182. We thus have arrived at the intuitively plausible picture that the density starts out sharply confined at x' at $t=0$, and then proceeds to spread; at some time $t>0$ we have



as t increases, the width increases as $\sqrt{t}$. Since the total amount of material is constant $\Rightarrow$ peak falls.

Pictorially:



For an initial distribution which is not a δ-function the solution is obtained as before by writing

$$\boxed{\psi(x,t) = \int dx'\, G(x,x',t)\,\psi(x',0) = \int dx' \left\{ \frac{1}{\sqrt{4\pi t}}\, e^{-(x-x')^2/4t} \right\} \psi(x',0)} \quad (52)$$

GREEN'S FUNCTION FOR THE HOMOGENEOUS WAVE EQN.

We repeat the steps in the previous discussion to find the Green's function for the homogeneous wave equation

$$\Box\psi(\vec{x},t) = \nabla^2\psi(\vec{x},t) - \frac{\partial^2\psi(\vec{x},t)}{c^2\,\partial t^2} = 0 \qquad ; \; \tau \equiv ct \tag{1}$$

This can be written in the form $H\psi = -\ddot{\psi} = -\partial^2\psi/\partial\tau^2$; $-\nabla^2 \equiv H$ (2)

As before we assume that

$$\psi(\vec{x},\tau) = \sum_m A_m(\tau)\,\phi_m(\vec{x}) \qquad ; \qquad H\phi_m = \lambda_m\phi_m \tag{3}$$

Hence
$$0 = H\psi + \ddot{\psi} = \sum_m \Big\{ -A_m(\tau)\underbrace{\nabla^2\phi_m(\vec{x})}_{-\lambda_m\phi_m} + \ddot{A}_m(\tau)\,\phi_m(\vec{x}) \Big\} \tag{4}$$

$$= \sum_m \Big\{ A_m(\tau)\lambda_m + \ddot{A}_m(\tau) \Big\}\phi_m(\vec{x})$$

Since the $\phi_m(\vec{x})$ are linearly independent it follows that the $A_m(\tau)$ are solutions of

$$\ddot{A}_m(\tau) + \lambda_m A_m(\tau) = 0 \Rightarrow \boxed{A_m(\tau) = a_m e^{i\sqrt{\lambda_m}\,\tau} + b_m e^{-i\sqrt{\lambda_m}\,\tau}} \tag{5}$$

Note that there are now 2 sets of unknown constants, a_m & b_m, which is a consequence of now dealing with a differential equation which is 2ND order in t.

Since $H = -\nabla^2$ is Hermitian the λ_m are real, but they may be positive or negative:

$$\psi(\vec{x},\tau) = \sum_m \Big\{ a_m e^{i\sqrt{\lambda_m}\,\tau} + b_m e^{-i\sqrt{\lambda_m}\,\tau} \Big\}\phi_m(\vec{x}) \tag{6}$$

To determine the a_m and b_m we use the boundary conditions which assume that $\psi(\vec{x},0)$ and $\dot{\psi}(\vec{x},0)$ are given. We then have:

$$\psi(\vec{x},0) = \text{known} = \sum_m (a_m+b_m)\phi_m(\vec{x}) \; ; \; \dot{\psi}(\vec{x},0) = \text{known} = \sum_m \left(i\sqrt{\lambda_m}\,a_m - i\sqrt{\lambda_m}\,b_m\right)\phi_m(\vec{x}) \tag{7}$$

We can invert these equations to solve for a_m & b_m by multiplying by $\phi_n^*(\vec{x})$ and integrating. By inspection we have

$$a_m + b_m = \int d^3x\, \phi_m^*(\vec{x})\, \psi(\vec{x},0) \equiv I_1^{\,m} \tag{8}$$

$$i\sqrt{\lambda_m}\,(a_m - b_m) = \int d^3x\, \phi_m^*(\vec{x})\, \dot{\psi}(\vec{x},0) = (a_m - b_m) = \frac{1}{i\sqrt{\lambda_m}}\int d^3x\, \phi_m^*(\vec{x})\, \dot{\psi}(\vec{x},0) \equiv I_2^{\,m} \tag{9}$$

$$a_m = \tfrac{1}{2}\left(I_1^{\,m} + I_2^{\,m}\right) \qquad b_m = \tfrac{1}{2}\left(I_1^{\,m} - I_2^{\,m}\right) \tag{10}$$

Since we must now specify 2 pieces of information to give the initial conditions, it follows that the Green's function solution will have the form

$$\boxed{\psi(\vec{x},\tau) = \int d^3x'\, G_1(\vec{x},\vec{x}',\tau)\, \psi(\vec{x}',0) + \int d^3x'\, G_2(\vec{x},\vec{x}',\tau)\, \dot{\psi}(\vec{x}',0)} \tag{11}$$

In other words there is a Green's function which propagates forward in time each of the 2 initial pieces of information.

PROOF OF EQ. (11): $\psi(\vec{x},\tau) = \sum_m A_m(\tau)\, \phi_m(\vec{x}) = \sum_m \left\{ e^{i\sqrt{\lambda_m}\tau} a_m + e^{-i\sqrt{\lambda_m}\tau} b_m \right\} \phi_m(\vec{x})$ (12)

$$\therefore \psi(\vec{x},\tau) = \sum_m \left\{ e^{i\sqrt{\lambda_m}\tau} \cdot \underbrace{\tfrac{1}{2}(I_1^m + I_2^m)}_{\sim a_m} + e^{-i\sqrt{\lambda_m}\tau} \cdot \underbrace{\tfrac{1}{2}(I_1^m - I_2^m)}_{\sim b_m} \right\} \phi_m(\vec{x}) \tag{13}$$

$$= \frac{1}{2}\sum_m \left\{ I_1^m \left(e^{i\sqrt{\lambda_m}\tau} + e^{-i\sqrt{\lambda_m}\tau}\right) + I_2^m \left(e^{i\sqrt{\lambda_m}\tau} - e^{-i\sqrt{\lambda_m}\tau}\right) \right\} \phi_m(\vec{x}) \tag{14}$$

Hence: $$\psi(\vec{x},\tau) = \sum_m \left\{ I_1^m \cos\sqrt{\lambda_m}\,\tau + i\, I_2^m \sin\sqrt{\lambda_m}\,\tau \right\} \tag{15}$$

Next we insert the explicit forms of I_1^m & I_2^m into (15):

$$\psi(\vec{x},\tau) = \sum_m \Big\{ \cos\sqrt{\lambda_m}\,\tau \cdot \int d^3x'\, \phi_m^*(\vec{x}')\,\psi(\vec{x}',0)$$
$$+ \cancel{i}\sin\sqrt{\lambda_m}\,\tau \cdot \frac{1}{\cancel{i}\sqrt{\lambda_m}} \int d^3x'\, \phi_m^*(\vec{x}')\,\dot{\psi}(\vec{x}',0) \Big\}\,\phi_m(\vec{x}) \qquad (16)$$

This can be rewritten as:

$$\boxed{\psi(\vec{x},\tau) = \int d^3x' \Big\{ \Big[\sum_m \cos(\sqrt{\lambda_m}\,\tau)\,\phi_m^*(\vec{x}')\,\phi_m(\vec{x})\Big]\psi(\vec{x}',0) + \Big[\sum_m \frac{1}{\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}\,\tau)\,\phi_m^*(\vec{x}')\,\phi_m(\vec{x})\Big]\dot{\psi}(\vec{x}',0)\Big\}} \qquad (17)$$

Compare (17) & (11): We see that

$$\boxed{\begin{aligned} G_1(\vec{x},\vec{x}',\tau) &= \sum_m \cos(\sqrt{\lambda_m}\,\tau)\,\phi_m^*(\vec{x}')\,\phi_m(\vec{x}) \\ G_2(\vec{x},\vec{x}',\tau) &= \sum_m \frac{1}{\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}\,\tau)\,\phi_m^*(\vec{x}')\,\phi_m(\vec{x}) \end{aligned}} \qquad (18)$$

We see from Eq. (18) that the expression given previously in (11) ~~is~~ not a guess (or an <u>ansatz</u>) but is a deriveable consequence of the Green's function formalism.

<u>Study of the Green's Function Solutions:</u>

Since a Green's function is the solution corresponding to a δ-function input we expect that $G_{1,2}(\vec{x},\vec{x}',\tau)$ will themselves be solutions of the homogeneous wave equation. Stated another way, we see from (11), (17), (18) that if $\psi(\vec{x},\tau)$ is a solution of the homogeneous wave equation, then $G_{1,2}(\vec{x},\vec{x}',\tau)$ must also be, since the only $\vec{x},\tau$ dependence of $\psi(\vec{x},\tau)$ comes from $G_{1,2}(\vec{x},\vec{x}',\tau)$.

Consider first $\Box_x G_1(\vec{x}, \vec{x}', \tau) = (\nabla_x^2 - \frac{\partial^2}{\partial\tau^2}) G_1$

$$\Box_x G_1(\dots) = \sum_m \cos(\sqrt{\lambda_m}\tau)\, \phi_m^*(\vec{x}')\, \underbrace{\nabla_x^2 \phi_m(\vec{x})}_{-\lambda_m \phi_m(\vec{x})} - \sum_m (-)(\sqrt{\lambda_m})^2 \cos(\sqrt{\lambda_m}\tau)\, \phi_m^*(\vec{x}')\, \phi_m(\vec{x}) = 0 \checkmark \quad (19)$$

these cancel.

Similarly:

$$\Box_x G_2(\vec{x}, \vec{x}', \tau) = \sum_m \frac{1}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}\tau)\, \phi_m^*(\vec{x}')\, \underbrace{\nabla_x^2 \phi_m(\vec{x})}_{-\lambda_m \phi_m(x)} - (-\sqrt{\lambda_m})^2 \frac{1}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}\tau)\, \phi_m^*(\vec{x}')\, \phi_m(\vec{x}) = 0 \checkmark \quad (20)$$

these cancel.

Another property of the Green's function solutions $G_{1,2}(\vec{x}, \vec{x}', \tau)$ is

$$\boxed{\dot{G}_2(\vec{x}, \vec{x}', \tau) = G_1(\vec{x}, \vec{x}', \tau)} \quad (21)$$

This follows by inspection from (18). From (21) it also follows that

$$\boxed{\dot{G}_2(\vec{x}, \vec{x}', 0) = G_1(\vec{x}, \vec{x}', 0) = \sum_m \phi_m^*(\vec{x}')\, \phi_m(\vec{x}) = \delta^3(x - \vec{x}')} \quad (22)$$

COMPLETENESS RELATION

DETAILED FUNCTIONAL FORM OF THE GREEN'S FUNCTIONS $G_{1,2}(\vec{x}, \vec{x}', \tau)$:

As in the case of the diffusion equation we want to find the functional forms of $G_{1,2}(\dots)$ once we specify appropriate boundary conditions. As before we assume that the solutions $\phi_m(\vec{x})$ correspond to quantizing in a box of volume V so that

$$\phi_m(\vec{x}) = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{x}} \quad (23)$$

Imposing periodic boundary conditions we want $\pm i\vec{k}\cdot\frac{\vec{L}}{2} = i\pi(\text{integer})$ (24)

Hence

$$\vec{k} = \frac{2\pi}{L}\vec{m} \equiv \frac{2\pi}{L}(m_x, m_y, m_z) \quad (25)$$

From (25): $k^2 = \left(\frac{2\pi}{L}\right)^2 (m_x^2 + m_y^2 + m_z^2)$; $m_x, m_y, m_z = 0, \pm 1, \pm 2, \ldots$ (26)

Inserting the solution (23) into the explicit forms for $G_1(\vec{x}, \vec{x}', \tau)$ in (18) we have:

$$G_1(\vec{x}, \vec{x}', \tau) = \frac{1}{L^3} \sum_{k_m} \cos(k_m \tau)\, e^{i \vec{k}_m \cdot (\vec{x} - \vec{x}')} \qquad (27)$$

Using the discussion of p. 185 we can immediately go over to the continuum limit via

$$\frac{1}{L^3} \sum_{k_m} \cdots \longrightarrow \int \frac{d^3k}{(2\pi)^3} \cdots \qquad (28)$$

Hence

$$G_1(\vec{x}, \vec{x}', \tau) = \int \frac{d^3k}{(2\pi)^3} \cos(k\tau)\, e^{i \vec{k} \cdot (\vec{x} - \vec{x}')} \qquad (29)$$

Note that in the expression $\cos(k\tau)$, $k = \sqrt{|\vec{k}|^2} = |\vec{k}|$, where we have written $\sqrt{\lambda_m} \to k$. Similarly,

$$G_2(\vec{x}, \vec{x}', \tau) = \int \frac{d^3k}{(2\pi)^3} \frac{\sin(k\tau)}{k}\, e^{i \vec{k} \cdot (\vec{x} - \vec{x}')} \qquad (30)$$

We first evaluate G_2 and then use $G_1 = \dot{G}_2$ to obtain G_1. To evaluate (30) define $\vec{\rho} = (\vec{x} - \vec{x}')$ and $\rho = |\vec{\rho}|$. Note that we are evaluating $G_2(\vec{x}, \vec{x}', \tau)$ for some given (fixed) values of $\vec{x}$ and $\vec{x}'$, which remain fixed during the integration over $\vec{k}$. Hence $\vec{\rho}$ is a fixed vector when $\int d^3k \ldots$ is evaluated. We can then conveniently choose to define the $\vec{k}$-coordinate system so that the z-axis in $\vec{k}$-space lies along $\hat{\rho}$. Then

in k-space

$$\vec{k} \cdot \vec{\rho} = k\rho\cos\theta \qquad (31)$$

and

$$G_2(\vec{x}, \vec{x}', \tau) = \int \frac{d^3k}{(2\pi)^3} \frac{\sin k\tau}{k}\, e^{ik\rho\cos\theta} \qquad (32)$$

Writing G_2 out in detail we have:

$$G_2(\vec{x},\vec{x}',\tau) = \frac{1}{(2\pi)^3}\int_0^\infty dk\cdot k^2 \int_{-1}^{1} d(\cos\theta) \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \left(\frac{e^{ik\tau}-e^{-ik\tau}}{2ik}\right) e^{ik\rho\cos\theta} \qquad (33)$$

Integrating over $\cos\theta$ gives:

$$G_2(\vec{x},\vec{x}',\tau) = \frac{1}{(2\pi)^2}\int_0^\infty dk\cdot k^2 \left(\frac{e^{ik\tau}-e^{-ik\tau}}{2ik}\right)\cdot \frac{1}{ik\rho}\, e^{ik\rho(\cos\theta)}\Bigg|_{\cos\theta=-1}^{\cos\theta=+1} = \qquad (34)$$

$k \Leftrightarrow -k$

$$= \left(\frac{1}{2\pi}\right)^2 \int_0^\infty dk\cdot \cancel{k^2}\, \frac{\left(e^{ik\tau}-e^{-ik\tau}\right)\left(e^{ik\rho}-e^{-ik\rho}\right)}{-2\cancel{k^2}\rho} = \frac{-1}{8\pi^2\rho}\int_0^\infty dk \left\{ e^{ik(\tau+\rho)} + e^{-ik(\tau+\rho)} - e^{-ik(\tau-\rho)} - e^{ik(\tau-\rho)} \right\} \qquad (35)$$

$k \Leftrightarrow -k$

From (35) we see that each pair of terms connected by $\Leftrightarrow$ is symmetric under the interchange $k \Leftrightarrow -k$. This allows us to extend the limits of integration from $-\infty$ to $+\infty$. Since this multiplies the result by a factor of 2, this can be offset simply by retaining only one term from each pair:

$$G_2(\vec{x},\vec{x}',\tau) = \frac{-1}{8\pi^2\rho}\int_{-\infty}^{\infty} dk \left\{ e^{ik(\tau+\rho)} - e^{ik(\tau-\rho)} \right\} = \frac{-1}{4\pi\rho}\left\{ \delta(\rho+\tau) - \delta(\rho-\tau) \right\}$$

$$= \frac{-1}{4\pi\rho}\left\{ \delta\left(|\vec{x}-\vec{x}'|+ct\right) - \delta\left(|\vec{x}-\vec{x}'|-ct\right) \right\} \qquad (36)$$

Hence, altogether,

$$\boxed{G_2(\vec{x},\vec{x}',t) = \frac{1}{4\pi|\vec{x}-\vec{x}'|}\left\{ \delta\left(|\vec{x}-\vec{x}'|-ct\right) - \delta\left(|\vec{x}-\vec{x}'|+ct\right) \right\}} \qquad (37)$$

We note that since $|\vec{x}-\vec{x}'| \geq 0$ only the first δ-function contributes when $t>0$ and only the second contributes when $t<0$.

Collecting the previous results together we can write:

$$\boxed{G_2(\vec{x},\vec{x}',t) \equiv D(|\vec{x}-\vec{x}'|,t) = \begin{cases} \dfrac{1}{4\pi}\,\dfrac{\delta(|\vec{x}-\vec{x}'|-ct)}{|\vec{x}-\vec{x}'|} & ;\ t>0 \\[2ex] \dfrac{-1}{4\pi}\,\dfrac{\delta(|\vec{x}-\vec{x}'|+ct)}{|\vec{x}-\vec{x}'|} & ;\ t<0 \end{cases}} \qquad (38)$$

Note that $G_2(\vec{x},\vec{x}',t)$ is symmetric in $\vec{x}$ and $\vec{x}'$, as expected.

Using Eqs. (11), (27), & (38) we can then write:

$$\boxed{\psi(\vec{x},t) = \int d^3x' \left\{ D(|\vec{x}-\vec{x}'|,t)\,\dot{\psi}(\vec{x}',0) + \frac{\partial}{c\partial t} D(|\vec{x}-\vec{x}'|,t)\,\psi(\vec{x}',0) \right\} \quad ;\ \dot{\psi} = \partial\psi/c\partial t} \qquad (39)$$

The contributions to D from $\delta(|\vec{x}-\vec{x}'|-ct)$ is known as the <u>RETARDED GREEN'S FUNCTION</u>, because the effect felt at some point $\vec{x}$ at a time t arises from a disturbance which originated at $\vec{x}'$ at an <u>earlier</u> ($\equiv$ retarded) time:

$$t_{ret} = t - \frac{|\vec{x}-\vec{x}'|}{c} \qquad (40)$$

The other contribution to D, which is proportional to $\delta(|\vec{x}-\vec{x}'|+ct)$ is called the <u>ADVANCED GREEN'S FUNCTION</u>. In Feynman's covariant treatment of quantum electrodynamics (QED) these enter in a symmetric way:

$$\boxed{\text{Feynman Green's Function} = \tfrac{1}{2}\,(\text{Retarded} + \text{Advanced})} \qquad (41)$$

<u>Connection to Coulomb's Law:</u> Note that by starting from the homogeneous law we have derived Coulomb's law $(\sim (1/4\pi)\, 1/|\vec{x}-\vec{x}'|)$ without having to put this in directly. The underlying connection between the wave equation and Coulomb's law is that both are direct consequences of the fact that photons are massless.

GREEN'S FUNCTION FOR THE INHOMOGENEOUS WAVE EQN

Up to this point we have derived the Green's function for the homogeneous wave equation

$$H\psi(\vec{x},t) + \ddot{\psi}(\vec{x},t) = 0 \qquad (1)$$

Next we derive the Green's function for the <u>inhomogeneous</u> <u>wave</u> <u>equations</u>

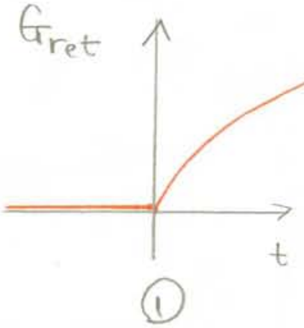
$$\boxed{H\psi(\vec{x},\tau) + \ddot{\psi}(\vec{x},\tau) = J(\vec{x},\tau)} \qquad (2)$$

where <u>J is a source current</u> and $\tau = ct$. $\psi(\vec{x},\tau)$ can denote either the scalar potential $\phi(\vec{x},t)$ or the vector potential $\vec{A}(\vec{x},t)$ whose source currents are $4\pi\rho(\vec{x},\tau)$ and $4\pi\vec{J}(\vec{x},\tau)$ respectively. Using the covariant notation from last semester we can express both of these cases in a single covariant notation:

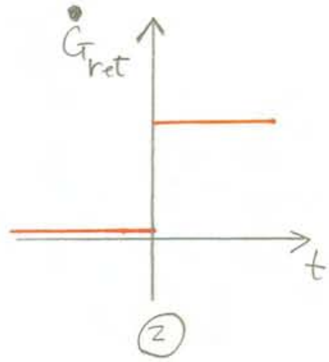
$$\Box A_\mu(\vec{x},\tau) = \frac{4\pi}{c} J_\mu(\vec{x},\tau) \;;\; A_\mu = (\vec{A},\phi) \qquad (3)$$
$$J_\mu = (\vec{J},\rho)$$

We assume that J_μ comes into existence at $t=0$, and hence we are looking for solutions for $t>0$. Pictorially the solution will have the following form:

We construct G_{ret} to have the form shown in ①: It is continuous at $t=0$ but its derivative is not (as in ②)

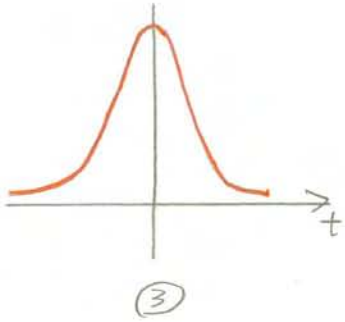


①



②

Because $\dot{G}_{ret}$ looks like a θ-function near the origin, its derivative $\ddot{G}_{ret}$ looks like a δ-function which is what we need for the inhomogeneous wave equation



③

Formal Derivation:

Using the results from the homogeneous wave equation we define a new Green's function $G_{ret}(\vec{x}, \vec{x}', \tau)$:

$$G_{ret}(\vec{x}, \vec{x}', \tau) = \begin{cases} G_2(\vec{x}, \vec{x}', \tau) & \tau > 0 \\ 0 & \tau < 0 \end{cases} \qquad (4)$$

Recall that $G_2(\vec{x}, \vec{x}', \tau) = \sum_m \frac{1}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}\,\tau)\, \phi_m^*(\vec{x}')\, \phi_m(\vec{x})$ (5)

We know from our previous work that $G_2(\vec{x}, \vec{x}', \tau)$ is a solution of the homogeneous wave equation so that

$$\boxed{\left(H + \frac{\partial^2}{\partial \tau^2}\right) G_{ret}(\vec{x}, \vec{x}', \tau) = \begin{cases} 0 & \tau > 0 \quad \text{(solves homogeneous eqn.)} \\ 0 & \tau < 0 \quad (\text{defined} = 0) \end{cases}} \qquad (6)$$

So the question then is <u>what happens to G_{ret} at $t=0$ when the source is "turned on"?</u>

To answer this question consider

$$I = \lim_{\epsilon \to 0_+} \int_{-\epsilon}^{\epsilon} d\tau \cdot \ddot{G}_{ret}(\vec{x}, \vec{x}', \tau) = \lim_{\epsilon \to 0_+} \dot{G}_{ret}(\vec{x}, \vec{x}', \tau) \Big|_{\tau=-\epsilon}^{\tau=\epsilon} \qquad (7)$$

$$= \lim_{\epsilon \to 0+} \underbrace{\dot{G}_{ret}(\vec{x}, \vec{x}', \tau)\Big|^{\tau=\epsilon}}_{\dot{G}_2 = G_1} - \lim_{\epsilon \to 0+} \underbrace{\dot{G}_{ret}(\vec{x}, \vec{x}', \tau)\Big|^{\tau=-\epsilon}}_{0} \qquad (8)$$

$\Rightarrow$ because we have defined $G_{ret} = 0$ for $\tau < 0$

Using the expression for $\dot{G}_2 = G_1$ from Eq.(22) pp.193,194 we have

$$I = \lim_{\epsilon \to 0_+} \sum_m \left\{ \cos(\sqrt{\lambda_m}\,\tau)\, \phi_m^*(\vec{x}')\, \phi_m(\vec{x}) \right\}^{\tau=\epsilon} = \lim_{\epsilon \to 0_+} \sum_m \cos(\sqrt{\lambda_m}\,\epsilon)\, \phi_m^*(\vec{x}')\, \phi_m(\vec{x}) \qquad (9)$$

$$= \sum_m \phi_m^*(\vec{x}')\, \phi_m(\vec{x}) = \delta^3(\vec{x} - \vec{x}')$$

Hence

$$\boxed{I = \lim_{\epsilon \to 0_+} \int_{-\epsilon}^{\epsilon} d\tau\, \ddot{G}_{ret}(\vec{x}, \vec{x}', \tau) = \delta^3(\vec{x} - \vec{x}')} \qquad (10)$$

Eq. (10) then allows us to write

$$\boxed{\ddot{G}_{ret}(\vec{x},\vec{x}',\tau) = \delta^3(\vec{x}-\vec{x}')\,\delta(\tau)} \quad (11)$$

This follows by noting that if we substitute (11) into (10) then

$$I = \lim_{\epsilon\to 0_+}\int_{-\epsilon}^{\epsilon} d\tau\; \delta^3(\vec{x}-\vec{x}')\,\delta(\tau) = \delta^3(\vec{x}-\vec{x}')\lim_{\epsilon\to 0^+}\underbrace{\int_{-\epsilon}^{\epsilon} d\tau\,\delta(\tau)}_{1} = \delta^3(\vec{x}-\vec{x}') \checkmark \quad (12)$$

Note that $\delta(\tau)$ is the only τ-dependent function which works in (11), for $\forall\,\epsilon$.

Using the above results we now return to Eq. (6) and consider

$$\left(H + \frac{\partial^2}{\partial\tau^2}\right)G_{ret} = H\,G_{ret}(\vec{x},\vec{x}',\tau) + \underbrace{\ddot{G}_{ret}(\vec{x},\vec{x}',\tau)}_{\delta^3(\vec{x}-\vec{x}')\,\delta(\tau) \;\leftarrow\; \text{from (11)}} = 0\begin{cases}\tau>0\\ \tau<0\end{cases} \quad (13)$$

From (6) we see that the r.h.s. of (13) $=0$ for both $\tau>0$ and $\tau<0$. For $\tau=0$ $\ddot{G}$ gives a contribution $\propto \delta(\tau)$, due to the discontinuity arising from the fact that the perturbation was "turned on" at $\tau=0$. However there is no additional contribution from $H\,G_{ret}$, since this involves spatial derivatives and there is no analogous spatial discontinuity. Hence altogether we find from (13):

$$\left(H + \frac{\partial^2}{\partial\tau^2}\right)G_{ret}(\vec{x},\vec{x}',\tau) = \left(-\nabla^2 + \frac{\partial^2}{c^2\partial t^2}\right)G_{ret}(\vec{x},\vec{x}',t) = \delta^3(\vec{x}-\vec{x}')\,\delta(t) \quad (14)$$

We can write the Green's function in a more symmetric way by shifting the τ coordinate (or the t coordinate) so that the perturbation is turned on at an arbitrary time t'. Then:

$$\boxed{\left(-\nabla^2 + \frac{\partial^2}{c^2\partial t^2}\right)G_{ret}(\vec{x},\vec{x}',\tau,\tau') = \delta^3(\vec{x}-\vec{x}')\,\delta(\tau-\tau') \equiv \delta^4(x-x')} \quad (15) \qquad \tau = ct$$

The fact that G_{ret} as defined in (4) satisfies (14) or (15) then means that it is the Green's function for the inhomogeneous wave equation.

From (14) & (15) we can verify that G_{ret} is a solution of the equation

$$\left(H + \frac{\partial^2}{\partial\tau^2}\right)\psi(\vec{x},\tau) = J(\vec{x},\tau) \qquad (16)$$

<u>Claim:</u> $\psi(\vec{x},\tau) \equiv \psi_{ret}(\vec{x},\tau) = \int d^3x'\, d\tau'\, G_{ret}(\vec{x},\vec{x}',\tau,\tau')\, J(\vec{x}',\tau')$ ✓ (17)

<u>Proof:</u>
$$\left(H+\frac{\partial^2}{\partial\tau^2}\right)\psi_{ret}(\vec{x},\tau) = \int d^3x'\, d\tau' \Big[\underbrace{\left(H+\frac{\partial^2}{\partial\tau^2}\right) G_{ret}(\vec{x},\vec{x}',\tau,\tau')}_{\delta^3(\vec{x}-\vec{x}')\,\delta(\tau-\tau')}\Big] J(\vec{x}',\tau') = J(\vec{x},\tau) \qquad (18)$$

It is convenient and instructive to write the Green's function G_{ret} explicitly, after letting $t \to t-t'$. Using (38) on p. 196, 197 we have:

$$G_2(\vec{x},\vec{x}',t,t') = \frac{1}{4\pi|\vec{x}-\vec{x}'|}\left\{\delta\left[|\vec{x}-\vec{x}'|-c(t-t')\right] - \delta\left[|\vec{x}-\vec{x}'|+c(t-t')\right]\right\} \qquad (19)$$

We note that for $(t-t')<0$ $G_{ret} \equiv 0$, so that the 2ND term in (19) makes no contribution, since it can only be non zero when $(t-t')<0$. Hence altogether

$$\psi_{ret}(\vec{x},t) = \int d^3x'\, \underbrace{d\tau'}_{c\,dt'} \left\{\frac{1}{4\pi|\vec{x}-\vec{x}'|}\,\delta\left[|\vec{x}-\vec{x}'|-c(t-t')\right]\right\} J(\vec{x}',t') \qquad (20)$$

Using the fact that $\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$ we can write:

$$\delta\left[|\vec{x}-\vec{x}'|-c(t-t')\right] = \frac{1}{c}\,\delta\left[\frac{|\vec{x}-\vec{x}'|}{c}-(t-t')\right] = \frac{1}{c}\,\delta\left[t'+\frac{|\vec{x}-\vec{x}'|}{c}-t\right] \qquad (21)$$

After cancelling the indicated factors of c we have:

$$\boxed{\psi_{ret}(\vec{x},t) = \frac{1}{4\pi}\int d^3x'\, dt' \left\{\frac{\delta\left[t'+\frac{|\vec{x}-\vec{x}'|}{c}-t\right]}{|\vec{x}-\vec{x}'|}\right\} J(\vec{x}',t')} \qquad (22)$$

We can more easily interpret this result if we first carry out the t' integration:

$$\boxed{\psi_{\text{ret}}(\vec{x},t) = \frac{1}{4\pi}\int d^3x' \, \frac{\left[J(\vec{x}',t')\right]_{\text{ret}}}{|x-\vec{x}'|}} \qquad (23)$$

In this expression $[\cdots]_{\text{ret}}$ means that we have used the δ-function in (22) to replace t' by

$$\boxed{t' \longrightarrow t - \frac{|\vec{x}-\vec{x}'|}{c}} \qquad (24)$$

We can then use (24) to write (23) explicitly as:

$$\boxed{\psi_{\text{ret}}(\vec{x},t) = \frac{1}{4\pi}\int d^3x' \, J\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\frac{1}{|\vec{x}-\vec{x}'|}} \qquad (25)$$

We see from (25) that an observer at $\vec{x}$ is affected at a time t only by what a source J at $\vec{x}'$ was doing at an <u>EARLIER TIME</u> $t - \frac{|\vec{x}-\vec{x}'|}{c}$, which accounts for the time that it takes an electromagnetic disturbance to propagate from $\vec{x}'$ to $\vec{x}$. This is the principle generally referred to as <u>CAUSALITY</u>