INTEGRAL EQUATIONS

Sometimes the solution to a problem naturally presents itself in the form of an integral equation where the unknown function is in the integrand. As an example

$$\phi(x) = \underset{\underset{\text{inhomogenous \underline{known} function}}{\uparrow}}{f(x)} + \underset{\underset{\text{constant}}{\uparrow}}{\lambda} \int_a^b dt \; \underset{\underset{\text{the "kernel" - \underline{known}}}{\uparrow}}{K(x,t)} \; \underset{\underset{\text{\underline{unknown} function}}{\uparrow}}{\phi(t)} \qquad (1)$$

An example follows below in which we show that one can convert a differential equation $Ly(x)=0$ along with its boundary conditions into an integral equation.

Consider the 1-dimensional Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = (\underset{\underset{\text{free Hamiltonian} = -(\hbar^2/2m)\nabla^2}{\uparrow}}{H_0} + \overset{\overset{\text{interaction Hamiltonian}}{\downarrow}}{H_I})\, \psi(\vec{x},t) \qquad (2)$$

This form of the Schrödinger equation is said to be in the SCHRÖDINGER PICTURE in which the wavefunction $\psi(\vec{x},t)$ is time-dependent, but operators such as $\vec{p}$ are not.

There is a 2ND picture in which the opposite is true: The operators are all time-dependent but the state vectors $|\psi\rangle$ are not. This is the HEISENBERG PICTURE. In this picture the time-dependence of an operator is given by

$$\dot{O}(t) = i\hbar\,[H,O] + \frac{\partial O}{\partial t} \longleftarrow \text{due to explicit time-dependence (e.g. } \vec{E} = \vec{E}_0 \cos\omega t) \qquad (3)$$

Hence even when there is no explicit time-dependence (ie when $\partial O/\partial t = 0$), it is still the case that $\dot{O}(t) \neq 0$.

There is yet a 3RD picture which is partway between the Schrödinger ~~equation~~ picture and the Heisenberg picture, which is called the DIRAC (or Interaction) PICTURE

In the Dirac picture the evolution of the state vector $|\psi(t)\rangle$ is governed by H_I, but the operators are governed by H_0:

$$\boxed{i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H_I|\psi(t)\rangle \;;\; \dot{O}(t) = i\hbar\,[H_0, O]} \qquad (4)$$

We will hencefort work in this picture in which the time-dependence of $|\psi(t)\rangle \equiv \psi(t)$ comes from H_I. We define an operator $\mathcal{U}(t_2,t_1)$ called the <u>TIME EVOLUTION OPERATOR</u> which has the following properties:

$$\psi(t_2) = \mathcal{U}(t_2,t_1)\,\psi(t_1) \qquad (5a)$$

Since the norm of $\psi(t) = \langle\psi(t)|\psi(t)\rangle = 1$, <u>$\mathcal{U}(t_2,t_1)$ must be Unitary</u>, $\mathcal{U}^+\mathcal{U} = 1$ (recall last semester). $\mathcal{U}(t_2,t_1)$ also satisfies:

$$\mathcal{U}(t_1,t_1) = \mathcal{U}(t_2,t_2) = 1 \qquad (5b)$$

$$\mathcal{U}(t_3,t_2)\,\mathcal{U}(t_2,t_1) = \mathcal{U}(t_3,t_1) \quad [\underline{\text{Group}}\ \underline{\text{Property}}] \qquad (5c)$$

Returning to the Schrödinger Equation in the interaction (Dirac) picture we can view the wavefunction at any time t as having evolved from the wavefunction $\psi(t_0)$ at some earlier time t_0:

$$\boxed{\psi(t) = \mathcal{U}(t,t_0)\,\psi(t_0)} \qquad (6)$$

Inserting Eq. (6) into Eq. (4) we have:

$$i\hbar \frac{\partial}{\partial t}\left\{\mathcal{U}(t,t_0)\,\psi(t_0)\right\} = H_I\left\{\mathcal{U}(t,t_0)\,\psi(t_0)\right\} \Rightarrow \qquad (7)$$

$$\left\{i\hbar \frac{\partial}{\partial t}\mathcal{U}(t,t_0)\right\}\cancel{\psi(t_0)} = \left\{H_I\,\mathcal{U}(t,t_0)\right\}\cancel{\psi(t_0)} \qquad (8)$$

Hence from (8) $\mathcal{U}(t,t_0)$ is a solution of

$$\boxed{\begin{gathered} i\hbar \frac{\partial}{\partial t}\mathcal{U}(t,t_0) = H_I(t,t_0)\,\mathcal{U}(\\ \mathcal{U}(t_0,t_0) = 1 \end{gathered}} \qquad (9)$$

We see that $\mathcal{U}(t,t_0)$ satisfies the same Schrödinger eqn as does $\psi(t)$. However the formulation in terms of $\mathcal{U}(t,t_0)$ has the advantage that it allows the boundary conditions to be specified at the same time in the form of the condition $\mathcal{U}(t_0,t_0)=1$. From (6) we see that solving for $\mathcal{U}(t,t_0)$ in terms of the b.c. is equivalent to solving the original Schrödinger equation.

To solve for $\mathcal{U}(t,t_0)$ in terms of H_I we note that we can formally integrate the Schrödinger equation in (9) [letting $t \to t_1$]:

$$\int_{t_0}^{t} dt_1 \frac{\partial}{\partial t_1}\mathcal{U}(t_1,t_0) = -\frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1)\,\mathcal{U}(t_1,t_0) \tag{10}$$

$$= \mathcal{U}(t_1,t_0)\Big|_{t_0}^{t} = \mathcal{U}(t,t_0) - \mathcal{U}(t_0,t_0) = \mathcal{U}(t,t_0) - 1 \quad \leftarrow \text{using (9)} \tag{11}$$

Hence:

$$\boxed{\mathcal{U}(t,t_0) = \underset{\substack{\uparrow \\ O(e^0)}}{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, \underset{\substack{\uparrow \\ O(e^1)}}{H_I(t_1)}\,\mathcal{U}(t_1,t_0)} \tag{12}$$

BORN APPROXIMATION

This is an integral equation for $\mathcal{U}(t,t_0)$: Note that $\mathcal{U}(t,t_0)$ sits both inside and outside the integral. The form of (12) suggests that $\mathcal{U}(t,t_0)$ can be solved for by an iterative procedure: Suppose that H_I is proportional to some small quantity like the electric charge e ($e^2 \simeq 1/137$). Hence successive powers of H_I introduce successively higher powers of e, so that many factors of $H_I \Rightarrow$ many powers of the small number e. Then, to lowest order:

$$\mathcal{U}(t,t_0) \cong 1 \tag{13}$$

We can then develop a series solution for $\mathcal{U}$ by writing:

$$\mathcal{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1)\,\mathcal{U}(t_1,t_0)$$

$$\longrightarrow 1 - \frac{i}{\hbar}\int_{t_0}^{t_1} dt_2\, H_I(t_2)\,\mathcal{U}(t_2,t_0) \tag{14}$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1)\left[1 - \frac{i}{\hbar}\int_{t_0}^{t_1} dt_2\, H_I(t_2)\,\mathcal{U}(t_2,t_0)\right] \tag{15}$$

$$= \underset{\substack{\uparrow \\ e^0}}{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, \underset{\substack{\uparrow \\ e^1}}{H_I}(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \underbrace{H_I(t_1)\, H_I(t_2)}_{\substack{\uparrow \\ e^2}}\,\mathcal{U}(t_2,t_0) \tag{16}$$

Hence to lowest non-trivial order:

$$\boxed{\mathcal{U}(t,t_0) \cong 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1)} \tag{17}$$

This procedure can obviously be repeated to generate a formal series solution for $\mathcal{U}(t,t_0)$. This solution is given by:

$$\mathcal{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0} dt_2\, H_I(t_1)\, H_I(t_2) + \cdots$$
$$+ \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1)\, H_I(t_2) \cdots H_I(t_n) \tag{18}$$

This is an example of what is called a Neumann-Series solution. The key question, of course, is whether this series converges [see below].

Classification of Integral Equations

[1] limits of integration	mnemonic
(a) fixed ⟷ Fredholm	$f \leftrightarrow F$
(b) variable ⟷ Volterra	$v \leftrightarrow V$

[2] Where the unknown appears

(a) under the integral only ⟷ equation of the 1ST kind

(b) outside the integral as well as under integral ⟷ equation of 2ND kind

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Specific Examples:

$$f(x) = \lambda \int_a^b dt\, K(x,t)\, \phi(t)$$ ⟷ Fredholm eqn. of the 1ST kind

$$\phi(x) = f(x) + \lambda \int_a^b dt\, K(x,t)\, \phi(t)$$ ⟷ Fredholm eqn. of the 2ND kind

$$f(x) = \lambda \int_a^x dt\, K(x,t)\, \phi(t)$$ ⟷ Volterra eqn. of the 1ST kind

$$\phi(x) = f(x) + \lambda \int_a^x dt\, K(x,t)\, \phi(t)$$ ⟷ Volterra eqn. of the 2ND kind

Terminology: $K(x,t)$ = kernel and is known. If $f(x)=0$ the integral eqn is said to be homogeneous; $f(x)$ is a known function. If $K(x,t) = K(t,x)$ then the kernel is said to be symmetric. If $K(x,t) = K_1(x)\, K_2(t)$ the kernel is said to be factorizable. The methods of solution of an integral equation depend sensitively on the nature of the kernel.

Methods of Solution of Integral Equations:

There are no all-encompassing methods for solving integral equations. We develop a collection of techniques which apply in different situations.

Solution by Integral Transforms:

known

unknown

Consider the equation: $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{ixt}\,\phi(t)$ FOURIER TRANSFORM (1)

This can be solved immediately via a Fourier TRANSFORM:

$$\boxed{\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-ixt}\, f(t)} \qquad (2)$$

Check:
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-ixt}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx'\, e^{ix't}\,\phi(x')\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\,\phi(x')\underbrace{\int_{-\infty}^{\infty} dt\, e^{it(x'-x)}}_{2\pi\delta(x-x')} = \phi(x) \checkmark \qquad (3)$$

We can understand the process of inverting (1) as merely a change of basis in Hilbert space. To see this consider the analogous discrete case as an example:

$$|f_i\rangle = \sum_j |\phi_j\rangle\langle\phi_j|f_j\rangle \leftarrow \text{solve for } \phi_j \qquad (4)$$

If f_i and g_j are orthonormal bases then we have: ← (5)

$$|f_i\rangle = \sum_j |g_j\rangle\langle g_j|f_i\rangle \Longleftrightarrow |g_j\rangle = \sum_k |f_k\rangle\langle f_k|g_j\rangle$$

kernel; this represents the "direction cosines" between 2 arbitrary bases $|f_i\rangle$ & $|g_j\rangle$

We can check the inversion in (5) as follows:

$$|f_i\rangle = \sum_j\left\{\sum_k |f_k\rangle\langle f_k|g_j\rangle\right\}\langle g_j|f_i\rangle = \sum_j\sum_k \underbrace{|f_k\rangle\langle f_k|}_{1}\underbrace{g_j\rangle\langle g_j|}_{1}f_i\rangle = |f_i\rangle \checkmark \qquad (6)$$

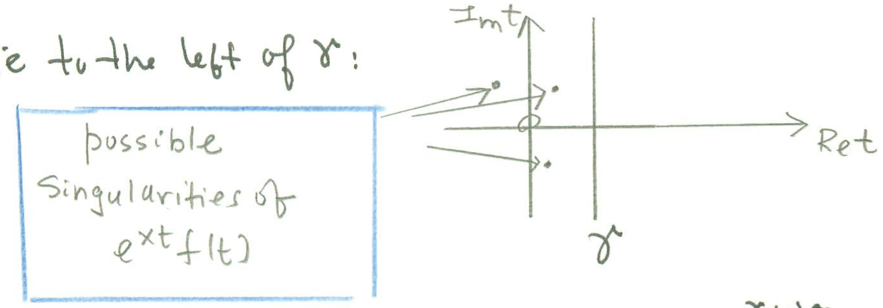
This verifies that (4) gives the correct prescription for going from one basis to another.

Note also that the coefficients $\langle g_j | f_i \rangle$ give the components R_{ij} of the rotation matrix connecting the two bases.

Other equations can also be solved by integral transforms:

LAPLACE TRANSFORMS: $$f(x) = \int_0^\infty dt\, e^{-xt}\,\phi(t) \longleftrightarrow \phi(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dt\, e^{xt} f(t) \qquad (7)$$

Here γ is a constant so chosen that all the singularities of $f(t)$ in the complex t-plane lie to the left of γ:



MELLIN TRANSFORM: $$f(x) = \int_0^\infty dt\, t^{x-1}\,\phi(t) \longleftrightarrow \phi(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dt\, x^{-t} f(t) \qquad (8)$$

HANKEL TRANSFORM: $$f(x) = \int_0^\infty dt\,[t J_\nu(xt)]\,\phi(t) \longleftrightarrow \phi(x) = \int_0^\infty dt\,[t J_\nu(xt)]\, f(t)$$

For later purposes we note that the kernels for the MELLIN & HANKEL transforms are not symmetric in x & t: $K(x,t) \neq K(t,x)$.

SOLUTION BY NEUMANN SERIES

The previous method using integral transforms has limited application. We turn to a general method using a series solution, which was developed by Liouville, Volterra, and Neumann. We will discuss it with reference to a generic Fredholm equation:

$$\longrightarrow \phi(t) = f(t) + \lambda \int_a^b dt_1\, K(t,t_1)\, \phi(t_1)$$

$$\underset{\substack{\uparrow\\ \text{unknown}}}{\phi(x)} = \underset{\substack{\uparrow\\ \text{known}}}{f(x)} + \underset{\substack{\uparrow\\ \text{known}\\ \text{constant}}}{\lambda} \int_a^b dt\, \underset{\substack{\uparrow\\ \text{known}}}{K(x,t)}\, \underset{\substack{\uparrow\\ \text{unknown}}}{\phi(t)} \qquad (1)$$

For the present we assume that λ is small (see below) – or that the series solution that we are about to derive converges for some other reason. Then from (1)

$$\phi(x) = f(x) + \lambda \int_a^b dt\, K(x,t) \left[f(t) + \lambda \int_a^b dt_1\, K(t,t_1)\, \phi(t_1) \right] \qquad (2)$$

$$\phi(x) = \underbrace{f(x) + \lambda \int_a^b dt\, K(x,t)/ f(t)}_{\text{known}} + \lambda^2 \int_a^b dt \int_a^b dt_1\, K(x,t)\, K(t,t_1)\, \phi(t_1) \qquad (3)$$

IMPORTANT!! Since $K(x,t) \neq K(t,x)$ in general, one must be very careful about the order of indices in $K(x,t)$ and $K(t,t_1)$ in (2) & (3)

We note from (3) that at this point $\phi(x)$ is determined through order λ^1 [that is we have determined $\phi(x)$ to $O(\lambda^0)$ and $O(\lambda^1)$] so that the remaining unknown contribution is at worst $O(\lambda^2)$. Repeating this process once more, we have (by inspection):

$$\phi(x) = f(x) + \lambda \int_a^b dt\, K(x,t) f(t) + \lambda^2 \int_a^b dt \int_a^b dt_1\, K(x,t)\, K(t,t_1)\, f(t_1) \longleftarrow \text{known}$$

$$+ \lambda^3 \int_a^b dt \int_a^b dt_1 \int_a^b dt_2\, K(x,t)\, K(t,t_1)\, K(t_1,t_2)\, \phi(t_2) \longleftarrow \text{unknown} \qquad (4)$$

Proceeding in this way we find that after n-iterations the coefficients of all terms through $O(\lambda^n)$ are determined, and are given by

$$[\text{Coefficient of } \lambda^n] = \int_a^b dt \cdots \int_a^b dt_{n-1}\, K(x,t)\cdots K(t_{n-2},t_{n-1})\, f(t_{n-1}) \qquad (5)$$

$$= \int_a^b dt_{n-1} \underbrace{\left\{ \int_a^b dt \int_a^b dt_1 \cdots \int_a^b dt_{n-2}\, K(x,t)\cdots K(t_{n-2},t_{n-1}) \right\}}_{K_n(x,t_{n-1})} f(t_{n-1}) \qquad (6)$$

$\equiv$ nth iterated kernel

We see that $K_n(x,t_{n-1})$ depends only on x and t_{n-1}, which are the only variables not being integrated over. [Note also that we have assumed that various integrations can be interchanged.] From Eqs. (4)-(6) we have:

$$\phi(x) = f(x) + \lambda \int_a^b dt \left\{ K_1(x,t) + \lambda K_2(x,t) + \cdots + \lambda^n K_{n+1}(x,t) + \cdots \right\} f(t) \qquad (7)$$

Note that in obtaining (7) we have renamed the last integration in each case by interchanging the labels t_{n-1} and t. Hence:

$$\boxed{\begin{array}{l} \phi(x) = f(x) + \lambda \int_a^b dt \left\{ \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) \right\} f(t) \qquad (8) \\ \qquad\qquad\qquad\qquad \swarrow \\ \phi(x) \equiv f(x) + \lambda \int_a^b dt\, \mathcal{K}(x,t;\lambda)\, f(t) \qquad (9) \\ \qquad\qquad\qquad \hookrightarrow \text{RESOLVENT KERNEL} \end{array}}$$

Note the formal similarity of the solution in (9) to the original integral equation:

$$\phi(x) = f(x) + \lambda \int_a^b dt\, K(x,t)\, \phi(t) \qquad (10)$$

Hence to solve (10) simply replace $K(x,t)$ by $\mathcal{K}(x,t;\lambda)$, and then replace the unknown function $\phi(t)$ by $f(t)$.

EQs (8) & (9) are a formal solution for $\phi(x)$. To ensure that this solution makes sense we have to worry about convergence of the sum in (8). We state without proof the following result: The series in (8) leads to a formally correct solution if $K(x,t)$ is square integrable which means that

$$\| K(x,t) \|^2 = \int_a^b dx \int_a^b dt \; K^2(x,t) < \infty \qquad (11)$$

Note that (contrary to the initial motivation!!) the convergence criterion does not depend on the magnitude of λ.

Uniqueness of Solutions: We consider the possible existence of 2 solutions $\phi_1(x)$ and $\phi_2(x)$ corresponding to the same $K(x,t)$ and the same $f(x)$, so that

$$\phi_1(x) = f(x) + \lambda \int_a^b dt\, K(x,t)\, \phi_1(t) \; ; \quad \phi_2(x) = f(x) + \lambda \int_a^b dt\, K(x,t)\, \phi_2(t) \qquad (12)$$

As usual, form the difference function $\omega(x) = [\phi_1(x) - \phi_2(x)] = \lambda \int_a^b dt\, K(x,t) [\phi_1(t) - \phi_2(t)]$

Hence $\omega(x)$ satisfies the homogeneous equation

$$\omega(x) = \lambda \int_a^b dt\, K(x,t)\, \omega(t) \qquad (13)$$

Then: $$\omega^2(x) = \lambda^2 \left[\int_a^b dt\, K(x,t) \omega(t) \right]^2 = \lambda^2 \int_a^b dt\, K(x,t) \omega(t) \cdot \int_a^b dt'\, K(x,t') \omega(t')$$

$$\leq \lambda^2 \int_a^b dt\, K(x,t) K(x,t) \cdot \underbrace{\int_a^b dt'\, \omega(t') \omega(t')}_{\text{positive constant}} \qquad (14)$$

The inequality in (14) is the SCHWARZ INEQUALITY in the form $(\vec{A} \cdot \vec{B})^2 \leq |\vec{A}|^2 |\vec{B}|^2$.

Here $A = K(x,t)$ and $B = \omega(t)$, so that

$$\vec{A} \cdot \vec{B} = \int_a^b dt\, K(x,t) \omega(t) \equiv \int_a^b dt\, K_x(t) \omega(t) \qquad (15)$$

← view x as an "index" to understand $\vec{A} \cdot \vec{B}$

Returning to (14) integrate both sides with respect to x:

$$\underbrace{\int_a^b dx\, \omega^2(x)}_{A^2=\text{constant}} \le \lambda^2 \underbrace{\int_a^b dx \int_a^b dt\ K(x,t)\, K(x,t)}_{\| K(x,t)\|^2} \cdot \underbrace{\int_a^b dt'\, \omega^2(t')}_{A^2=\text{constant}} \tag{16}$$

$$\text{Eq. (16)} \Rightarrow \left(1-\lambda^2 \|K\|^2\right) \int_a^b dx\, \omega^2(x) \le 0 \tag{17}$$

However, in the original integral equation <u>λ is an arbitrary constant</u>; hence for fixed $\|K\|^2$ we can always choose λ small enough so that $(1-\lambda^2\|K\|^2)>0$. It follows that the only way to satisfy (17) is if

$$A^2 = \int_a^b dx\, \omega^2(x) = 0 \Rightarrow \omega(x)=0 \Rightarrow \boxed{\phi(x_1)=\phi(x_2) \quad \text{Q.E.D.}} \tag{18}$$

<u>Side Comments:</u> (a) From the preceding discussion it follows that if we start with a <u>homogeneous</u> equation

$$\omega(x) = \lambda \int_a^b dt\ K(x,t)\, \omega(t) \tag{19}$$

the only way to obtain a non-trivial solution [i.e. different from $\omega(x)\equiv 0$] is if $\lambda^2\|K\|^2 \geqslant 1$.

(b) It also follows from (17) that in the <u>inhomogeneous</u> case, if $\lambda^2\|K\|^2>1$ then the inequality in (17) can hold even though $\omega(x)\neq 0$. This suggests that solutions $\phi_1(x)$ and $\phi_2(x)$ can be found with $\phi_1 \neq \phi_2$.

EXAMPLE: Solve $\phi(x) = f(x) + \lambda \int_0^1 dt\, \underbrace{e^{x-t}}_{K(x,t)}\, \phi(t)$ (1)

SOLUTION: Compute the RESOLVENT KERNEL $\mathcal{K}(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$ (2)

$$n=0 \Rightarrow K_1(x,t) = K(x,t) = e^{x-t} \qquad (3)$$

$$n=1 \Rightarrow K_2(x,t) = \int_0^1 dt_1\, K(x,t_1)\, K(t_1,t) = \int_0^1 dt_1\, e^{x-t_1}\, e^{t_1-t} = \int_0^1 dt_1\, e^{x-t} \qquad (4)$$

note the importance of keeping the order of indices correct!!

$$= e^{x-t}\int_0^1 dt_1 = e^{x-t} = K(x,t) \qquad (5)$$

$$n=2 \Rightarrow K_3(x,t) = \int_0^1 dt_1 \int_0^1 dt_2\, \underbrace{e^{x-\cancel{t_1}}\, e^{\cancel{t_1-t_2}}\, e^{\cancel{t_2}-t}}_{e^{x-t}} = e^{x-t}\int_0^1 dt_1 \int_0^1 dt_2 = e^{x-t} = K(x,t) \qquad (6)$$

$\vdots$

Hence all of the nth iterated kernels are equal, which allows us to write in (2):

$$\mathcal{K}(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) = K(x,t)\sum_{n=0}^{\infty} \lambda^n = K(x,t)\,\frac{1}{1-\lambda} \Leftarrow \text{if } |\lambda|<1$$
$$= e^{x-t}\,(1/1-\lambda) \qquad (7)$$

Hence using 214(8) & (9) we find:

$$\phi(x) = f(x) + \lambda\int_0^1 dt\, \mathcal{K}(x,t;\lambda) f(t) = f(x) + \lambda\int_0^1 dt \left\{\frac{e^{x-t}}{1-\lambda}\right\} f(t) \qquad (8)$$

Finally:

$$\boxed{\phi(x) = f(x) + \frac{\lambda e^x}{1-\lambda}\int_0^1 dt\, e^{-t} f(t)} \qquad (9)$$

We note for future reference that the steps leading from (7) to (9) are valid only if $|\lambda|<1$.

Note also that since $\overset{=c'}{\int_0^1 dt \cdots}$ = constant, we know that the functional dependence of $\phi(x)$ must be: $\boxed{\phi(x) = f(x) + c'\left(\frac{\lambda}{1-\lambda}\right) e^x}$ (10)

ORDER OF ARGUMENTS OF $K(x,t)$ etc.

Consider the integral equation on p. 217

$$\phi(x) = f(x) + \lambda \int_0^1 dt \; K(x,t)\,\phi(t) \qquad K(x,t) = e^{x-t} \qquad (1)$$

Following the discussion on p. 213 the solution to $O(\lambda^2)$ is given by

$$\phi(x) = f(x) + \lambda \int_0^1 dt\, K(x,t) f(t) + \lambda^2 \int_0^1 dt \int_0^1 dt_1\, K(x,t)\, K(t,t_1)\, f(t_1) \qquad (2)$$

where we have replaced $\phi(t_1) \to f(t_1)$ in 213(3), valid to $O(\lambda^2)$

Using (1) this then gives:

$$\phi(x) = f(x) + \lambda \int_0^1 dt\, e^{x-t} f(t) + \lambda^2 \int_0^1 dt \int_0^1 dt_1\, e^{x-\cancel{t}}\, e^{\cancel{t}-t_1}\, f(t_1) \qquad (3)$$

$$= f(x) + \lambda e^x \int_0^1 dt\, e^{-t} f(t) + \lambda^2 \int_0^1 dt \int_0^1 dt_1\, e^{x-t_1}\, f(t_1) \qquad (4)$$

$$= f(x) + \lambda e^x \int_0^1 dt\, e^{-t} f(t) + \lambda^2 e^x \int_0^1 dt_1\, e^{-t_1} f(t_1) \underbrace{\int_0^1 dt}_{1} \qquad (5)$$

Since t_1 in (5) is now a dummy variable, we can rename it $t_1 \to t$ at this stage, so the expression in (5) assumes the form

$$\phi(x) = f(x) + (\lambda e^x + \lambda^2 e^x) \int_0^1 dt\, e^{-t} f(t) \qquad (6)$$

This agrees exactly with the result in 217(9) to $O(\lambda^2)$. ✓

The connection to the method of the RESOLVENT KERNEL, is seen in (4): At this stage, both t and t_1 are dummy integration variables, so that we can interchange them in (4) to give

$$\phi(x) = f(x) + \lambda e^x \int_0^1 dt\, e^{-t} f(t) + \lambda^2 e^x \int_0^1 dt\, e^{-t} f(t) \underbrace{\int_0^1 dt_1}_{1} \qquad (7)$$

This interchange is what we referred to on p. 214 just above Eq. (8): We can always rename the integration variables (recognizing that they are dummy variables) in such a way that <u>the last integration is carried out on the variable t as in (6) and (7).</u>

This step is what allows us to write the solution as in 214(8):

$$\phi(x) = f(x) + \lambda \int_a^b dt \left\{ \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) \right\} f(t)$$

As might be expected, when the solution can be found this easily, it can also be found by another method which does not require the use of resolvent kernels. This leads us to the case of DEGENERATE KERNELS.

Definition: A kernel is said to be SEPARABLE or DEGENERATE if

$$\boxed{K(x,t) = \sum_{j=1}^{n} M_j(x) N_j(t)} \qquad (11)$$

where the sum is finite. As an example consider the trigonometric function $\cos(t-x)$:

$$\cos(t-x) = \cos t \cos x + \sin t \sin x \qquad (12)$$

The kernel in the previous example was separable because $e^{x-t} = e^x e^{-t}$. Knowing this we could have arranged to solve the previous integral equation as follows:

$$\phi(x) = f(x) + \lambda \int_0^1 dt\, e^{x-t} \phi(t) = f(x) + \lambda e^x \underbrace{\int_0^1 dt\, e^{-t} \phi(t)}_{\text{Constant} \equiv C} = f(x) + \lambda C e^x \qquad (13)$$

$$\text{Then } C = \int_0^1 dt\, e^{-t} \phi(t) = \int_0^1 dt\, e^{-t} \left[f(t) + \lambda C e^t\right] \qquad (14)$$

$$\therefore C = \int_0^1 dt\, e^{-t} f(t) + \lambda C \int_0^1 dt = \int_0^1 dt\, e^{-t} f(t) + \lambda C \qquad (15)$$

Collecting terms: $C(1-\lambda) = \int_0^1 dt\, e^{-t} f(t) \Rightarrow \boxed{C = \frac{1}{1-\lambda} \int_0^1 dt\, e^{-t} f(t)}$ (16)

Combining (13) & (16) then gives:

$$\boxed{\phi(x) = f(x) + \lambda C e^x = f(x) + \frac{\lambda}{1-\lambda} e^x \int_0^1 dt\, e^{-t} f(t)} \qquad (17)$$

As expected this agrees completely with the result obtained in (9) via the RESOLVENT KERNEL method. Note, however, that the restrictions $|\lambda| < 1$ which was needed there does not apply here! This leads us to conclude that the solution in (9) could have been analytically continued to $|\lambda| > 1$.

The question now remains as to what happens when $\lambda = 1$?
Note that both methods break down here. From (9) or (17) we see that in this case we can find a sensible solution only when

$$\int_0^1 dt\, e^{-t} f(t) = 0 \qquad (18)$$

Since this cannot be true in general for arbitrary $f(t)$ it follows that for $\lambda = 1$ there is no solution to the integral equation in general [for all $f(t)$].

However, when this condition is met we claim that the solution to the original integral equation is

$$\boxed{\phi(x) = f(x) + \underset{\uparrow\text{arbitrary constant}}{A} e^{x}} \qquad (19)$$

Before proving this we note that we could have gotten to this result by observing from (17) that $\phi(x)$ has the functional form

$$\phi(x) = f(x) + \lambda C e^{x} \quad ; \ C = \text{known} \qquad (20)$$

Not surprisingly, when $\lambda = 1$ we cannot determine C, so that λC gets replaced by an arbitrary constant A, which leads to (19). To prove that (19) is in fact a solution we write (when $\lambda = 1$)

$$\phi(x) = f(x) + 1\cdot e^{x} \int_0^1 dt\, e^{-t} \underset{\hookrightarrow f(t) + Ae^{t}}{\phi(t)} \longrightarrow f(x) + e^{x} \underbrace{\int_0^1 dt\, e^{-t} f(t)}_{= 0 \text{ using (18)}} + A e^{x} \underbrace{\int_0^1 dt\, \cancel{e^{-t}}\, \cancel{e^{t}}}_{= 1} \qquad (21)$$

$$\therefore\ \phi(x) = f(x) + Ae^{x} \text{ is a solution!} \qquad = f(x) + Ae^{x}\ \checkmark$$

Note that (18) can hold even when $\lambda \neq 1$ in which case we find from (9) or (17) that the solution to the original integral equation is

$$\phi(x) = f(x) \qquad (22)$$

General Method for Separable Kernels:

We consider the general case of a Kernel which has the form

$$K(x,t) = \sum_{j=1}^{n} M_j(x)\, N_j(t) \qquad (1)$$

Then for a general integral equation of the form $\phi(x) = f(x) + \lambda \int_a^b dt\, K(x,t)\phi(t)$ we have

$$\phi(x) = f(x) + \lambda \sum_{j=1}^{n} M_j(x) \underbrace{\int_a^b dt\, N_j(t)\,\phi(t)}_{\text{Constant} \equiv C_j} = f(x) + \lambda \sum_{j=1}^{n} C_j M_j(x) \qquad (2)$$

So we know that the functional form of $\phi(x)$ must be

$$\boxed{\phi(x) = f(x) + \lambda \sum_{j=1}^{n} C_j M_j(x)} \qquad (3)$$

In principle this gives a solution for $\phi(x)$ except that the C_j – which themselves depend on $\phi(x)$ – are not yet known. To determine them multiply (3) by $N_i(x)$ and integrate w.r.t. x:

$$\underbrace{\int_a^b dx\, N_i(x)\phi(x)}_{= C_i} = \underbrace{\int_a^b dx\, N_i(x) f(x)}_{= b_i \text{ (known)}} + \lambda \sum_{j=1}^{n} C_j \underbrace{\int_a^b dx\, N_i(x) M_j(x)}_{= a_{ij} \text{ (known)}} \qquad (4)$$

Hence

$$\boxed{C_i = b_i + \lambda \sum_j a_{ij} C_j} \qquad (5) \qquad \text{Let } \vec{\vec{A}} = (a_{ij})$$

$$\hookrightarrow \vec{C} = \vec{b} + \lambda \vec{\vec{A}}\vec{c} \quad \Rightarrow \text{ dropping } \Rightarrow, \rightarrow \quad C - \lambda A C = b \qquad (6)$$

Hence as a matrix equation

$$\boxed{(I - \lambda A)C = b} \qquad (7)$$

We can formally solve this equation for C if $(I - \lambda A)^{-1}$ exists:

$$\boxed{C = (I - \lambda A)^{-1} b} \qquad (8)$$

Eqs. (7) & (8) lead to a system of algebraic equations which can be solved simultaneously for the c_i. For example, starting directly from Eq. (7) we can write:

$$(1-\lambda a_{11})c_1 + (-\lambda a_{12})c_2 + (-\lambda a_{13})c_3 + \cdots = b_1 \qquad (9)$$

$$(-\lambda a_{21})c_1 + (1-\lambda a_{22})c_2 + (-\lambda a_{23})c_3 + \cdots = b_2$$

$$\vdots$$

If the integral is <u>homogeneous</u> [i.e. $f(x)=0$] then from Eq. (4) $b_i \equiv 0$. Then from (7) the c_i are solutions to the equation

$$\boxed{(I-\lambda A)c = 0} \qquad c \equiv \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} \qquad (10)$$

From our previous discussions of linear algebra we recall that Eq. (10) would have only a trivial (and here nonsensical!) solution $C = (c_1, c_2, \ldots) = 0$ unless $(I-\lambda A)^{-1}$ did not exist, which is equivalent to demanding that

$$\boxed{\det(I-\lambda A) = 0} \qquad (11)$$

As usual solving this equation first gives the eigenvalues λ_i. Then when these are inserted into the original equations (10) the λ_i lead to a determination of the c_i as well.

<u>Example:</u> Solve $\phi(x) = \int_{-1}^{1} dt \underbrace{(x+t)}_{K(x,t)} \phi(t)$ (12)

This can be written as the following degenerate kernel:

$$K(x,t) = \sum_{j=1}^{2} M_j(x) N_j(t) = \underset{\substack{\downarrow \\ M_1(x)}}{x} \cdot \underset{\substack{\downarrow \\ N_1(t)}}{1} + \underset{\substack{\downarrow \\ M_2(x)}}{1} \cdot \underset{\hookrightarrow N_2(t)}{t} \qquad (13)$$

$$\text{Then: } a_{ij} = \int_{-1}^{1} dx\, N_i(x) M_j(x) \Rightarrow a_{11} = \int_{-1}^{1} dx\, \overset{N_1(x)}{1} \cdot \underset{M_1(x)}{x} = \tfrac{1}{2} x^2 \Big|_{-1}^{1} = 0 \qquad (14)$$

$$= a_{22} \text{ (by symmetry)}$$

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Similarly: $a_{12} = \int_{-1}^{1} dx\, N_1(x) M_2(x) = \int_{-1}^{1} dx\, 1\cdot 1 = 2$ (15)

$$a_{21} = \int_{-1}^{1} dx\, N_2(x) M_1(x) = \int_{-1}^{1} dx\, x\cdot x = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}$$

Hence $A = (a_{ij}) = \begin{bmatrix} 0 & 2 \\ \frac{2}{3} & 0 \end{bmatrix}$ (16)

It follows that the eigenvalue equation (11) becomes

$$0 = \det(I - \lambda A) = \det\begin{pmatrix} 1 & -\lambda\cdot 2 \\ -\lambda\cdot\frac{2}{3} & 1 \end{pmatrix} = 1 - \lambda^2 \frac{4}{3} = 0 \Rightarrow \boxed{\lambda = \pm\frac{\sqrt{3}}{2}} \qquad (17)$$

We next insert (17) into (10) so that (10) reads:

$$(I - \lambda A)\begin{pmatrix} c_1^{\pm} \\ c_2^{\pm} \end{pmatrix} = 0 = \begin{pmatrix} 1 & \mp\frac{\sqrt{3}}{2}\cdot 2 \\ \mp\frac{\sqrt{3}}{2}\cdot\frac{2}{3} & 1 \end{pmatrix}\begin{pmatrix} c_1^{\pm} \\ c_2^{\pm} \end{pmatrix} = \begin{pmatrix} 1 & \mp\sqrt{3} \\ \frac{\mp 1}{\sqrt{3}} & 1 \end{pmatrix}\begin{pmatrix} c_1^{\pm} \\ c_2^{\pm} \end{pmatrix} \qquad (18)$$

$$= \begin{pmatrix} c_1 \mp \sqrt{3}\, c_2 \\ \mp\frac{1}{\sqrt{3}} c_1 + c_2 \end{pmatrix} = 0 \qquad \begin{matrix} \longrightarrow c_1 = \pm\sqrt{3}\, c_2 \\ \longrightarrow c_2 = \pm\frac{1}{\sqrt{3}} c_1 \Rightarrow c_1 = \pm\sqrt{3}\, c_2 \end{matrix} \Bigg\} \text{same solutions} \qquad (19)$$

Since the original integral equation is <u>homogeneous</u> the overall scale of $\phi(x)$ is irrelevant, and hence we can arbitrarily set $\boxed{c_1 = +1 \Rightarrow c_2 = \pm\frac{1}{\sqrt{3}}}$ (20)

The solutions to the integral equation are, then:

$$\lambda = +\frac{\sqrt{3}}{2} \Rightarrow c_1 = 1;\ c_2 = \frac{+1}{\sqrt{3}} \Rightarrow \phi(x) = \frac{\sqrt{3}}{2}\left[1\cdot M_1(x) + \frac{1}{\sqrt{3}} M_2(x)\right] = \frac{\sqrt{3}}{2}\left[1\cdot x + \frac{1}{\sqrt{3}}\cdot 1\right] = \frac{\sqrt{3}}{2}\left[x + \frac{1}{\sqrt{3}}\right]$$

$$\lambda = -\frac{\sqrt{3}}{2} \Rightarrow c_1 = 1;\ c_2 = \frac{-1}{\sqrt{3}} \Rightarrow \phi(x) = \left[1\cdot x - \frac{1}{\sqrt{3}}\cdot 1\right] = -\frac{\sqrt{3}}{2}\left[x - \frac{1}{\sqrt{3}}\right]$$

Hilbert-Schmidt Theory:

This is the theory of Symmetric kernels: $K(x,t) = K(t,x)$ (1)

To start with, a symmetric kernel can sometimes be formed out of a kernel which does not look symmetric initially. Consider, as an example

$$\phi(x) = f(x) + \lambda \int_a^b dt\,[K(x,t)\underset{\uparrow\ \text{known function}}{\rho(t)}]\,\phi(t) \tag{2}$$

Here we assume $K(x,t) = K(t,x)$; however, the function which plays the role of the kernel is $K(x,t)\rho(t)$ which is not manifestly symmetric. However, suppose that we multiply (2) by $\sqrt{\rho(x)}$, then:

$$\sqrt{\rho(x)}\,\phi(x) = \sqrt{\rho(x)}\,f(x) + \lambda \int_a^b dt\,[K(x,t)\,\rho(t)\sqrt{\rho(x)}\,]\,\phi(t) \tag{3}$$

$$\underbrace{\sqrt{\rho(x)}\,\phi(x)}_{\equiv\psi(x)} = \sqrt{\rho(x)}\,f(x) + \lambda \int_a^b dt\,[K(x,t)\sqrt{\rho(t)}\,\sqrt{\rho(x)}\,]\,\underbrace{\sqrt{\rho(t)}\,\phi(t)}_{\psi(t)} \tag{4}$$

Hence $\psi(x)$ satisfies:

$$\boxed{\psi(x) = \sqrt{\rho(x)}\,f(x) + \lambda \int_a^b dt\,[K(x,t)\underbrace{\sqrt{\rho(x)}\,\sqrt{\rho(t)}}_{\text{Symmetric}}\,]\,\psi(t)} \tag{5}$$

Clearly once we have solved for $\psi(x)$ we can obtain $\phi(x)$ as $\phi(x) = \psi(x)/\sqrt{\rho(x)}$.

Consider first a Fredholm equation of the first kind:

$$\boxed{\phi(x) = \lambda \int_a^b dt\ K(x,t)\,\phi(t)} \tag{6}$$

If $K(x,t)$ is continuous, it can be shown that there is at least one eigenvalue λ which solves this equation. (We will not prove this here.)

We next show that λ must be real if $K(x,t)$ is symmetric, and also that the solutions corresponding to different λ_i are orthogonal. Write,

$$\phi_i(x) = \lambda_i \int_a^b dt\, \underbrace{K(x,t)}_{\text{real}}\, \phi_i(t) \qquad\qquad \phi_j(x) = \lambda_j \int_a^b dt\, \underbrace{K(x,t)}_{\text{real}}\, \phi_j(t) \qquad (7)$$

$\otimes\, \lambda_j \phi_j$ $\qquad\qquad$ $\otimes\, \lambda_i \phi_i$

Multiplying and integrating gives:

$$\lambda_j \int_a^b dx\, \phi_i(x)\phi_j(x) = \lambda_i\lambda_j \int_a^b dx \int_a^b dt\; K(x,t)\,\phi_i(t)\,\phi_j(x) \qquad (8)$$

$$\lambda_i \int_a^b dx\, \phi_j(x)\phi_i(x) = \lambda_j\lambda_i \int_a^b dx \int_a^b dt\; K(x,t)\,\phi_j(t)\,\phi_i(x) \qquad (9)$$

$$= \lambda_j\lambda_i \int_a^b dx \int_a^b dt\; K(x,t)\,\underset{\sim\sim\sim\sim}{\phi_j(x)\,\phi_i(t)} \qquad (10)$$

Explanation of $\sim\sim\sim$: Here we use the fact that $K(x,t)$ is symmetric and $\int dx \int dt$ is also symmetric (in $x \leftrightarrow t$). Hence $\phi_j(x)\phi_i(t)$ must also be symmetric in $x \leftrightarrow t$ in order to contribute to the integral. Specifically we can write:

$$\phi_j(t)\,\phi_i(x) = \underbrace{\tfrac{1}{2}\left[\phi_j(t)\phi_i(x) + \phi_j(x)\phi_i(t)\right]}_{\text{SYMMETRIC IN } (x \leftrightarrow t)} + \underbrace{\tfrac{1}{2}\left[\phi_j(t)\phi_i(x) - \phi_j(x)\phi_i(t)\right]}_{\text{ANTI-SYMMETRIC IN } (x \leftrightarrow t)} \qquad (11)$$

Clearly the anti-symmetric part in (11) integrates to zero, so that only the symmetric part contributes, with each term contributing equally (by symmetry). This is what allows us to go from (9) to (10). Having (10), we can now subtract (10) from (8), noting that the r.h.s. of these two equations are equal. It then follows immediately that

$$\boxed{(\lambda_i - \lambda_j)\int_a^b dx\, \phi_i(x)\phi_j(x) = 0 = (\lambda_i - \lambda_j)\langle \phi_i | \phi_j \rangle \;;\; \text{hence } \lambda_i \neq \lambda_j \Rightarrow \underbrace{\langle \phi_i | \phi_j \rangle = 0}_{\text{ORTHOGONALITY}}} \qquad (12)$$

Note that for a symmetric kernel no complex conjugation is needed. This is not the case for the related case of Hermitian kernels.

If two eigenfunctions $\phi_i(x)$ and $\phi_j(x)$ have the same eigenvalue λ, then these eigenfunctions are said to be degenerate. In that case the previous discussion has to be modified: We can use the GRAM-SCHMIDT method to form out of ϕ_i and ϕ_j 2 orthonormal eigenfunctions.

To show that λ_i is real, we begin with Eqs.(6),(7) taking complex conjugation, but using $K(x,t)=K^*(x,t)$

$$\phi_i^*(x) = \lambda_i^* \int_a^b dt\, K(x,t)\, \phi_i^*(t) \qquad (13)$$

We now use this equation in place of the equation $\phi_j(x) = \dots$ in (7). Proceeding as before we find immediately

$$(\lambda_i^* - \lambda_i) \int_a^b dx\; \phi_i^*(x)\, \phi_i(x) = 0 = \int_a^b dx\, |\phi_i(x)|^2 \qquad (14)$$

Now of course $\int \dots$ cannot vanish unless $\phi_i(x) \equiv 0$, trivially. It then follows that $\boxed{\lambda_i^* = \lambda_i \Rightarrow \lambda_i \text{ is REAL},}$ Q.E.D. (15)

The fact that the eigenvalues are real for a symmetric kernel $K(x,t)$, means that we can expect such integral equations to play an important role in quantum mechanics, and to be related to Hermitian Operators.

It can also be shown that the solutions $\phi_i(x)$ of the integral equations form a complete set. Using this fact we infer that we can expand any function, such as the kernel $K(x,t)$ itself in terms of these eigenfunctions:

$$K(x,t) = \sum_{n=1}^{\infty} a_n(x)\, \phi_n(t) \qquad (16)$$

$\uparrow$ CON set of eigenfunctions

As usual we can determine the expansion coefficients $a_n(x)$ by multiplying (16) on the left by $\phi_m(t)$ and integrating:

$$\int_a^b dt\, K(x,t)\phi_m(t) = \sum_{n=1}^{\infty} a_n(x) \underbrace{\int_a^b dt\, \phi_m(t)\phi_n(t)}_{\delta_{mn}} = a_m(x) \qquad (17)$$

Hence: $$a_m(x) = \underbrace{\int_a^b dt\, K(x,t)\phi_m(t)}_{\lambda_m^{-1}\phi_m(x)} \qquad (18)$$

$\lambda_m^{-1}\phi_m(x)$ ← using the original integral equation

Hence combining (16) & (18) we find: $$\boxed{K(x,t) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(t)}{\lambda_n} \quad ; \lambda_n \neq 0} \qquad (19)$$

Note that in this form $K(x,t)$ is manifestly symmetric. Note there are cases when this expansion does not work, for example when $\lambda_n = 0$ for some n.

Note that we have discussed the properties of integral equations when $K(x,t) = K(t,x)$, and not the actual solutions. To actually solve these equations, the previous methods can be used.

Solutions of the Inhomogeneous Equation for $K(x,t)=K(t,x)$

Consider: $$\phi(x) = f(x) + \lambda\int_a^b dt\, K(x,t)\,\phi(t) \qquad (1)$$

We assume that the solutions of the homogeneous equation are known:

$$\phi_n(x) = \lambda_n \int_a^b dt\, K(x,t)\,\phi_n(t) \qquad (2)$$

As previously we can solve the inhomogeneous equation by expanding both $\phi(x)$ and $f(x)$ in terms of the known eigenfunctions $\phi_n(x)$. Thus:

$$\phi(x) = \sum_{n=1}^{\infty} a_n\phi_n(x) \qquad\qquad f(x) = \sum_{n=1}^{\infty} b_n\phi_n(x) \qquad (3)$$

Inserting (3) into (1) we find

$$\sum_{n=1}^{\infty} a_n\phi_n(x) = \sum_{n=1}^{\infty} b_n\phi_n(x) + \underbrace{\lambda\int_a^b dt\, K(x,t)\sum_{n=1}^{\infty} a_n\phi_n(t)}_{\lambda\sum_n a_n \underbrace{\int_a^b dt\, K(x,t)\,\phi_n(t)}_{\lambda_n^{-1}\phi_n(x)}} \quad \begin{array}{l}\leftarrow \text{interchanging } \Sigma \,\&\, \int \\ \leftarrow \text{using (2)}\end{array} \qquad (4)$$

$$\text{Hence}\quad \sum_{n=1}^{\infty} a_n\phi_n(x) = \sum_n b_n\phi_n(x) + \lambda\sum_n \left(\frac{a_n}{\lambda_n}\right)\phi_n(x) \qquad (5)$$

Since the functions $\phi_n(x)$ are linearly independent, Eq.(5) holds only if

$$a_n = b_n + \frac{\lambda}{\lambda_n}a_n \Rightarrow \boxed{a_n = \frac{\lambda_n b_n}{\lambda_n - \lambda} = \frac{b_n}{1-\frac{\lambda}{\lambda_n}}} \qquad (6)$$

Then: $$\boxed{\phi(x) = \sum_n a_n\phi_n(x)} \qquad (7)$$

We can use the result in (6) to obtain another expression for $\phi(x)$ by writing

$$\phi(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{\lambda_n b_n}{\lambda_n - \lambda} \phi_n(x) = \sum_{n} \left\{ \frac{\lambda_n - \lambda + \lambda}{\lambda_n - \lambda} b_n \right\} \phi_n(x) \quad (8)$$

$$= \sum_{n=1}^{\infty} \left\{ \left(1 + \frac{\lambda}{\lambda_n - \lambda}\right) b_n \right\} \phi_n(x) = \underbrace{\sum_{n=1}^{\infty} b_n \phi_n(x)}_{f(x)} + \lambda \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n - \lambda}\right) b_n \phi_n(x) \quad (9)$$

Now b_n can be read off by inspection from (3):

$$b_n = \int_a^b dt\, f(t)\, \phi_n(t) \quad (10)$$

Combining (9) & (10):

$$\boxed{\phi(x) = f(x) + \lambda \sum_{n=1}^{\infty} \left\{ \left(\frac{1}{\lambda_n - \lambda}\right) \int_a^b dt\, f(t)\, \phi_n(t)\, \phi_n(x) \right\}} \quad (11)$$

We note that both terms in (11) are proportional to $f(t)$. Hence if $f(x)=0$ then either $\phi(x)=0$ trivially or else $\lambda = \lambda_i$ for some λ_i. In this latter case both the numerator and denominator in $\Sigma\cdots$ vanish, but in such a manner that their quotient may be finite and nonzero.

Thus the only nontrivial solution of the homogeneous equation holds when $\lambda = \lambda_i$, an eigenvalue.

What happens when $\lambda = \lambda_p$ (an eigenvalue)?

Returning to Eq. (6) above we have: $a_p = b_p + \frac{\lambda}{\lambda_p} a_p \rightarrow b_p + \frac{\cancel{\lambda_p}}{\cancel{\lambda_p}} a_p$ (12)

Hence $a_p = b_p + a_p \Rightarrow \boxed{b_p = 0}$ and $\boxed{a_p = \text{undetermined}}$ (13)

From (10) above we see that $b_p = \int_a^b dt\, f(t)\phi_p(t)$ and hence the only way that this integral equation has a solution is if $\boxed{b_p = \int_a^b dt\, f(t)\phi_p(t) = 0}$ (14)

Otherwise there is no solution!

Even when there is no solution for b_p, for some p, the general solution in (11) holds for $n \neq p$. Hence we can write:

$$\sum_{n=1}^{\infty} = \sum_{\substack{n=1 \\ n\neq p}}^{\infty}{}' + (\text{Contribution from } n=p) \qquad (15)$$

Hence we can write for $\phi(x)$ (using (11) above)

$$\boxed{\phi(x) = f(x) + \underset{\text{undetermined}}{a_p}\,\phi_p(x) + \lambda \sum_{\substack{n=1 \\ n\neq p}}^{\infty}{}' \left\{ \left(\frac{1}{\lambda_n - \lambda} \right) \int_a^b dt\, f(t)\,\phi_n(t)\,\phi_n(x) \right\}} \qquad (16)$$

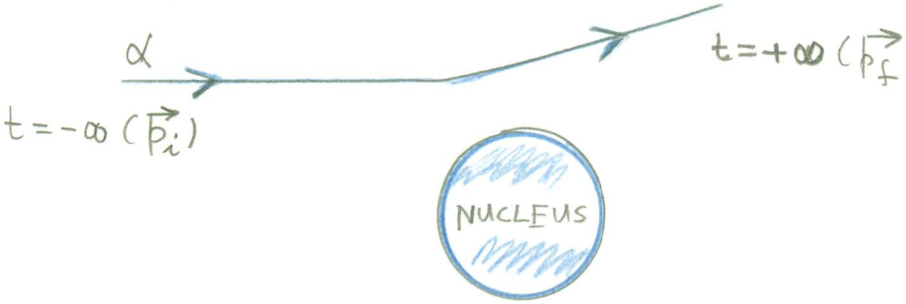
INTEGRAL EQUATIONS, THE S-MATRIX & PERTURBATION THEORY

Referring to p. 208 we see that $\mathcal{U}(t,t_0)$ can be solved for in an iterative manner by writing

$$\mathcal{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H_I(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) + \dots \tag{1}$$

$$+ \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1) H_I(t_2) \dots H_I(t_n) + \dots$$

When H_I describes a <u>scattering process</u>, such as the scattering of an α-particle off a nucleus (<u>Rutherford Scattering</u>),



then as a practical matter we can picture the α-particle as incident at a time $t_0 = -\infty$, which is then detected (after scattering) at a later time $t = +\infty$. Hence what we are interested in in this case is

$$\boxed{S \equiv \mathcal{U}(+\infty, -\infty)} = 1 - \left(\frac{i}{\hbar}\right)\int_{-\infty} dt_1\, H_I(t_1) +$$

$$+ \left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2\, H_I(t_1) H_I(t_2) + \left(\frac{-i}{\hbar}\right)^3 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3\, H_I(t_1) H_I(t_2) H_I(t_3) + \dots$$

$$+ \left(\frac{-i}{\hbar}\right)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n\, H_I(t_1) \dots H_I(t_n) + \dots = S^{(0)} + S^{(1)} + S^{(2)} + \dots S^{(n)} \tag{2}$$

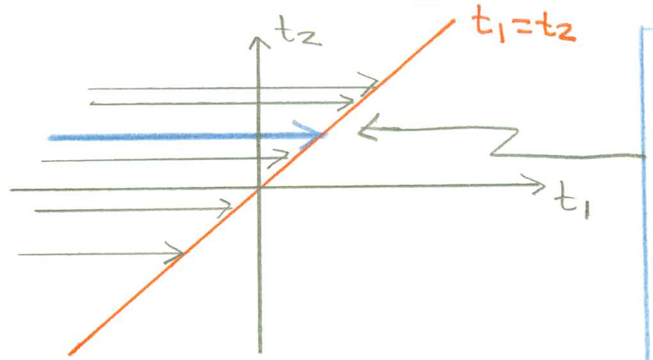
It is convenient to rewrite (2) in the following way:

Consider the contribution $S^{(2)}$

$$S^{(2)} = \left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \, H_I(t_1) H_I(t_2) \tag{3}$$

$$= \left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \, H_I(t_2) H_I(t_1) \tag{4}$$

where we have used the fact that t_1 & t_2 are <u>dummy variables</u> of integration to interchange $t_1 \longleftrightarrow t_2$. Graphically the integration region looks as follows:

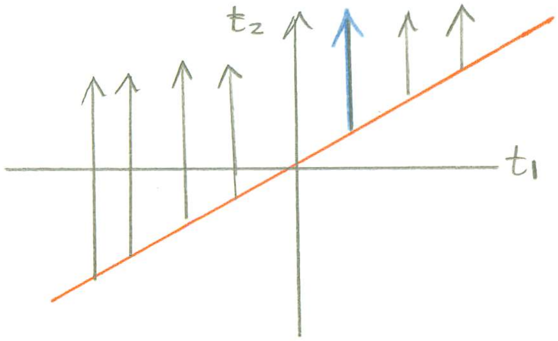


First integrate (horizontally) over t_1 from $-\infty$ to t_2. Then add up all these contributions by integrating (vertically) over t_2 from $-\infty$ to $+\infty$.

It follows from this figure that the region of integration comprises exactly half of the t_1-t_2 plane. Since it should make no difference how we integrate over this half-plane it follows that (3) is equivalent to

$$S^{(2)} = \left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \, H_I(t_2) H_I(t_1) \tag{5}$$

Note that the integrands in (4) and (5) are the same: $H_I(t_2) H_I(t_1)$. To see the equivalence of (3) & (5) we exhibit the regions of integration in (5):



First integrate (vertically) over t_2 (for t_1 fixed) from t_1 to $+\infty$. Then integrate (horizontally) over t_1 from $-\infty$ to $+\infty$.

We see from (5) and the figure that the region of integration in (5) is the same half-plane that contributes in (4), and hence (5) and (4) are identical. Since (4) and (3) are identical it then follows that (5) and (3) are also identical. It then follows that we can write for $S^{(2)}$

$$S^{(2)} = \frac{1}{2}\left[(3)+(5)\right] = \left(\frac{-i}{\hbar}\right)^2 \left\{ \frac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \, H_I(t_1)H_I(t_2) + \frac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \, H_I(t_2)H_I(t_1) \right\} \quad (6)$$

The two contributions in (6) can be combined by introducing the DYSON CHRONOLOGICAL OPERATOR P which works as follows:

$$P\, H_I(t_1)\, H_I(t_2) = \begin{cases} H_I(t_1)\, H_I(t_2) & \text{if } t_1 > t_2 \\ H_I(t_2)\, H_I(t_1) & \text{if } t_2 > t_1 \end{cases} \quad (7)$$

Simply stated, P arranges operators so that the operator with the earliest time argument is placed all the way to the right, etc. Then (6) & (7) ⇒

$$\boxed{S^{(2)} = \left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \; P\left\{ H_I(t_1)\, H_I(t_2) \right\}} \quad (8)$$

To verify (8) we note that if we evaluate $\int dt_2$ for some fixed (finite) value of t_1 then initially $t_1 > t_2$, so $P\{\ldots\}$ leaves the operators in $\{\ldots\}$ unchanged, and this reproduces the first term in (6). Eventually, $t_2 > t_1$ is reached, and then $P\{\ldots\}$ replaces $\{\ldots\}$ in (8) by $\{H_I(t_2)H_I(t_1)\}$, which then reproduces the second terms in (6).

Evidently the same procedure can be applied to all the terms $S^{(n)}$ in Eq. (2) so that we can write

$$S = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n \; P\left\{ H_I(t_1)\, H_I(t_2) \cdots H_I(t_n) \right\} \quad (9)$$

If we define the Hamiltonian density $\mathcal{H}_I(\vec{x},t) \equiv \mathcal{H}_I(x)$ such that

$$H_I(t) = \int d^3x\, \mathcal{H}_I(\vec{x},t) = \int d^3x\, \mathcal{H}_I(x) \qquad (10)$$

then

$$\boxed{S = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \cdots \int_{-\infty}^{\infty} d^4x_n \; P\left\{\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)\ldots\mathcal{H}_I(x_n)\right\} \quad (11)}$$

One often sees this sum formally written as $(d^4x \equiv d^3x\,dt)$

$$S = P\left\{ e^{-(i/\hbar)\int d^4x\, \mathcal{H}_I(x)} \right\} \qquad (12)$$

In this form S is manifestly covariant under Lorentz transformations.

BERNOULLI NUMBERS & BERNOULLI FUNCTIONS

9/4/84

Derivation of the Euler-MacLaurin Formula:

Ref. ARFKEN, Mathematical Methods for Physicists p.278 ff; Whittaker & Watson, Analysis, p.125 ff

We begin by introducing the Bernoulli numbers B_n via the generating function

$$\boxed{\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}} \qquad (1)$$

We can obtain the Bernoulli numbers by noting from the rhs of (1) that

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = B_0 + B_1 x + B_2 \frac{1}{2!} x^2 + \cdots \Rightarrow B_0 = \left[\sum B_n \frac{x^n}{n!}\right]_{x=0} \qquad (2)$$

$$\text{But} \quad \left.\frac{x}{e^x-1}\right|_{x=0} = \frac{0}{0} \rightarrow \text{l'Hopital's Rule to} \quad \left.\frac{1}{e^x}\right|_{x=0} = 1 \qquad (3)$$

$$\therefore \quad \boxed{B_0 = 1} \checkmark \qquad (4)$$

$$\text{Similarly:} \quad B_1 = \frac{d}{dx}\left(\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\right)_{x=0} = \frac{d}{dx}\left(\frac{x}{e^x-1}\right)_{x=0} = \frac{(e^x-1)\cdot 1 - xe^x}{(e^x-1)^2} \rightarrow \frac{0}{0} \qquad (5)$$

$$\text{Again by l'Hopital's rule} \quad B_1 \rightarrow \frac{\cancel{e^x} - xe^x - \cancel{e^x}}{2(e^x-1)} \rightarrow \frac{0}{0} \qquad (6)$$

$$\text{Still Again by l'Hopital:} \quad \boxed{B_1 \rightarrow \frac{-xe^x - e^x}{2e^x} = -\frac{1}{2}} \checkmark \qquad (7)$$

As can be seen, the repeated derivatives will lead to the formula [note that the $n!$ factors cancel]

$$\boxed{B_n = \frac{d^n}{dx^n}\left(\frac{x}{e^x-1}\right)_{x=0}} \checkmark \qquad (8)$$

To avoid the repeated derivatives we can cross multiply through in Eq. (1) to get:

$$1 = \frac{(e^x - 1)}{x} \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{(\cancel{1} + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \cancel{-1})}{x} \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (9)$$

$$\boxed{\therefore \quad 1 = \left(1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \dots + \frac{1}{k!}x^{k-1}\right)\left(B_0 + B_1 x + B_2 \frac{x^2}{2!} + \dots\right)} \qquad (10)$$

Constant term: $\underline{\underline{1}} = B_0$ ✓

x^1: $0 = \frac{1}{2!} B_0 + B_1$ $\Bigg\} \Rightarrow \underline{\underline{B_1 = -\frac{1}{2}}}$ ✓ (11)

x^2: $0 = \frac{1}{2!} B_2 + \underset{\checkmark}{\frac{1}{2!} B_1} + \frac{1}{3!} B_0 = \frac{1}{2} B_2 + \frac{1}{2} B_1 + \frac{1}{6} B_0$

$$\therefore \quad B_2 + B_1 + \frac{1}{3} B_0 = 0 \Rightarrow B_2 = -B_1 - \frac{1}{3} B_0 = +\frac{1}{2} - \frac{1}{3} = \underline{\underline{\frac{1}{6}}} \ \checkmark \qquad (12)$$

x^3: $0 = \frac{1}{3!} B_3 + \frac{1}{2!}\frac{1}{2!} B_2 + \frac{1}{3!} B_1 + \frac{1}{4!} B_0$

$$\therefore B_3 = -\frac{6}{4} B_2 - B_1 - \frac{6}{24} B_0 = -\frac{3}{2} \cdot \frac{1}{6} + \frac{1}{2} - \frac{1}{4} = 0 \ \checkmark \qquad (13)$$

5! = 120
4! = 24
3! = 6

x^4: $0 = \frac{1}{4!} B_4 + \frac{1}{2!}\frac{1}{3!} B_3 + \frac{1}{3!}\frac{1}{2!} B_2 + \frac{1}{4!} B_1 + \frac{1}{5!} B_0$

$$B_4 = -24\left(\frac{1}{12} \underset{0}{\underset{\parallel}{B_3}} + \frac{1}{12} B_2 + \frac{1}{24} B_1 + \frac{1}{120} B_0\right) = -\frac{24}{12} \cdot 0 - 2B_2 - B_1 - \frac{1}{5} B_0$$

$$= -2\frac{1}{6} + 1/2 - \frac{1}{5} = -\frac{10}{30} + \frac{15}{30} - \frac{6}{30} = -\frac{1}{30} \ \checkmark \qquad (14)$$

Arfken notes that <u>the odd B_n for $n \geq 3$ are all zero</u>, but this isn't obvious at the moment:

$$\boxed{B_{2n+1} \equiv 0 \qquad n \geq 1} \qquad (15)$$

BERNOULLI FUNCTIONS & REVIEW OF GENERATING FUNCTIONS

Bernoulli Functions:

We can introduce the Bernoulli function by writing

Generating function for $B_n(s)$

$$\boxed{\frac{x}{e^x-1}\cdot e^{xs} \equiv \sum_{n=0}^{\infty} B_n(s)\,\frac{x^n}{n!}} \qquad (16)$$

Clearly the Bernoulli functions and the Bernoulli numbers are related by

$$B_n(0) = B_n \quad ; \quad B_0(s) \equiv 1 \qquad (17)$$

The Bernoulli functions $B_n(s)$ have the following 2 important properties. First, differentiating (16) w.r.t. s we have

$$\frac{d}{ds}\left(\frac{x e^{xs}}{e^x-1}\right) = \frac{x^2 e^{xs}}{e^x-1} = \sum_{n=0}^{\infty} B_n'(s)\,\frac{x^n}{n!} \quad \Big\} \text{ Cancel a factor of } x\text{:} \qquad (18)$$

$$\therefore\ \frac{x e^{xs}}{e^{x}-1} = \sum_{n=0}^{\infty} B_n'(s)\,\frac{x^{n-1}}{n!} \quad \Big\}\ \underline{\text{look at the term of } x^{n-1}}\text{; Its coeff is } \frac{B_n'(s)}{n!} \qquad (19a)$$

$$\hookrightarrow = \sum_{n=0}^{\infty} B_n(s)\,\frac{x^n}{n!} \quad \Big\}\ \text{look at the term of } x^{n-1}\text{: Its coeff is } \frac{B_{n-1}(s)}{(n-1)!} \qquad (19b)$$

From (19a,b) we then have

$$\frac{B_n'(s)}{n!} = \frac{B_{n-1}(s)}{(n-1)!} \Rightarrow \boxed{B_n'(s) = n\,B_{n-1}(s)}\ \checkmark \qquad (20)$$

Consider next [redacted] $S=1 \Rightarrow$

$$\frac{x e^{x}}{e^x-1} = \frac{x}{1-e^{-x}} = -\frac{(-x)}{1-e^{-x}} = \frac{(-x)}{e^{(-x)}-1} = \sum_{n=0}^{\infty} B_n\,\frac{(-x)^n}{n!} \qquad (21)$$

$\therefore S=1 \Rightarrow$

$$\sum_n B_n(1)\frac{x^n}{n!} = \ldots\ldots = \sum_n (-1)^n B_n\,\frac{x^n}{n!} \Rightarrow \boxed{B_n(1) = (-1)^n B_n = (-1)^n B_n(0)} \qquad (22)\ \checkmark$$

THE EULER-MACLAURIN FORMULA:

Derivation of the E-M Formula:

Start with $\int_0^1 dx\, f(x) = \int_0^1 dx\, B_0(x)\, f(x)$ (28)

Note: By inspection $B_0(s) = 1 \quad \forall s$ [See (17)] (29)

Here the argument of B_n is x :

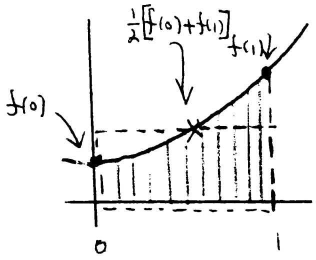
Then $B_1'(x) = 1 \cdot B_0(x) = 1$ [Using (20)] (30)

$$\therefore \int_0^1 dx\, f(x) = \int_0^1 dx\, B_1'(x)\, f(x) = B_1(x) f(x)\Big]_0^1 - \int_0^1 dx\, B_1(x)\, f'(x) \quad (31)$$

$$(22) \Rightarrow \quad = f(1)\underbrace{\underbrace{B_1(1)}_{1/2}}_{B_1(1) = (-1) B_1(0) = -B_1 = +1/2} - f(0)\underbrace{B_1(0)}_{-1/2} = \tfrac{1}{2}\left[f(1) + f(0)\right] \quad (32)$$

$$\boxed{\therefore \int_0^1 dx\, f(x) = \tfrac{1}{2}\left[f(1) + f(0)\right] - \int_0^1 dx\, B_1(x)\, f'(x) \;\checkmark} \quad (33)$$

This formula makes obvious sense from the point of view of basic calculus: It represents the most naive approximation to the calculation of an integral, as shown by the following figure:



The shaded area is then approximated by the area of the rectangle ¦‾‾‾¦ which is

$$\text{Area} \approx \text{base} \times \text{height} = 1 \cdot \tfrac{1}{2}\left[f(0) + f(1)\right] \quad (34)$$

If $f(x) = a$ then this would be the exact expression for the area, which is Consistent with (33), since then $f'(x) = 0$.

Continuing as before we use (20) to write

(20) ⇒
$$\boxed{B_2'(x) = 2B_1(x) \qquad \Rightarrow B_1(x) = \tfrac{1}{2}B_2'(x)} \qquad (35)$$

$$\Downarrow$$

$$\therefore \int_0^1 dx\, f(x) = \tfrac{1}{2}[f(0)+f(1)] - \int_0^1 dx\, \tfrac{1}{2} B_2'(x)\, f'(x)$$

$$= \tfrac{1}{2}[f(0)+f(1)] - \tfrac{1}{2}\underbrace{B_2(x) f'(x)\Big|_0^1}_{} + \tfrac{1}{2}\int_0^1 dx\, B_2(x) f''(x) \qquad (36)$$

$$-\tfrac{1}{2}[B_2(1) f'(1) - B_2(0) f'(0)]$$

We can invoke the general results:

$$\boxed{\begin{aligned} &B_{2n}(1) = (-1)^{2n} B_{2n}(0) = B_{2n} \\ &B_{2n+1}(1) = -B_{2n+1} = 0 \quad \text{for } n \geq 1 \\ &B_{2n}(0) = B_{2n} \\ &B_{2n+1}(0) = B_{2n+1} = 0 \end{aligned}} \qquad (37)$$

$$\Downarrow (36)$$

$$\therefore B_2(1) = B_2 = \tfrac{1}{6} = B_2(0) \Rightarrow -\tfrac{1}{2}[\dots] = -\tfrac{1}{2}\cdot\tfrac{1}{6}[f'(1) - f'(0)] \qquad (38)$$

$$\therefore \int_0^1 dx\, f(x) = \tfrac{1}{2}[f(0)+f(1)] - \tfrac{1}{2}\tfrac{1}{6}[f'(1) - f'(0)] + \tfrac{1}{2}\int_0^1 dx\, B_2(x) f''(x) \qquad \checkmark\ (39)$$

If we find a situation in which the higher derivatives are small [because as in our case each successive derivative introduces a factor of Δ'] then we can stop with the first derivatives by writing

$$\boxed{\int_0^1 dx\, f(x) = \tfrac{1}{2}[f(0)+f(1)] - \tfrac{1}{12}[f'(1) - f'(0)] + \tfrac{1}{2}\int_0^1 dx\, B_2(x) f''(x)} \qquad (40)$$

We can proceed to write down the general expression as in (5.168a) of ARFKEN

$$\boxed{\int_0^1 dx\, f(x) = \tfrac{1}{2}[f(1)+f(0)] - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(1) - f^{(2p-1)}(0)\right] + \frac{1}{(2q)!}\int_0^1 dx\, f^{(2q)}(x)\, B_{2q}(x)} \qquad (41)$$

DEFINITE INTEGRALS WITH ARBITRARY LIMITS:

Changing the limits of integration

Thus far the results in (40) and (41) hold specifically for the range of integration $[0,1]$. This is obvious if we recall that we have used the specific values of the Bernoulli polynomials for these values, $B_n(1)$ and $B_n(0)$. However, we now wish to extend the limits of integration first of all from $[0,\infty]$ and ultimately between any two limits. To do this let us for illustrative [redacted] purposes consider (40) neglecting the remainder:

$$\therefore \quad \int_0^1 dx\, f(x) \cong \frac{1}{2}\left[f(x=1)+f(x=0)\right] - \frac{1}{12}\left[f'(x=1)-f'(x=0)\right] + \dots \qquad (42)$$

Let the indefinite integral $\int f(x)\,dx \equiv F(x)$. Then (42) reads as

$$F(x=1)-F(x=0) \cong \frac{1}{2}\left[f(x=1)+f(x=0)\right] - \frac{1}{12}\left[f'(x=1)-f'(x=0)\right] + \dots \qquad (43)$$

Define $y = x+1$; Then (43) ⇒

$$\underbrace{F(y=2) - F(y=1)}_{\equiv \int_1^2 dy\, f(y)} \cong \frac{1}{2}\left[f(y=2)+f(y=1)\right] - \frac{1}{12}\left[f'(y=2)-f'(y=1)\right] \qquad (44)$$

Since y and x are now dummy variables of integration we see that by shifting in this way we can write

$$\boxed{\int_1^2 dx\, f(x) \cong \frac{1}{2}\left[f(2)+f(1)\right] - \frac{1}{12}\left[f'(y=2)-f'(y=1)\right] + \dots} \qquad (45)$$

We can continue this process, and doing so let us write down the first few terms in the series

$$\int_0^4 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx$$

$$\begin{aligned}
= &\ \frac{1}{2}\left[f(1)+f(0)\right] - \frac{1}{12}\left[\cancel{f'(1)}-f'(0)\right] + \frac{1}{2}\int_0^1 dx\, B_2(x)\, f''(x) \\
&+ \frac{1}{2}\left[f(1)+f(2)\right] - \frac{1}{12}\left[\cancel{f'(2)}-\cancel{f'(1)}\right] + \frac{1}{2}\int_1^2 dx\, B_2(x)\, f''(x) \\
&+ \frac{1}{2}\left[f(2)+f(3)\right] - \frac{1}{12}\left[\cancel{f'(3)}-\cancel{f'(2)}\right] + \frac{1}{2}\int_2^3 dx\, B_2(x)\, f''(x) \\
&+ \frac{1}{2}\left[f(3)+f(4)\right] - \frac{1}{12}\left[f'(4)-\cancel{f'(3)}\right] + \frac{1}{2}\int_3^4 dx\, B_2(x)\, f''(x)
\end{aligned} \qquad (46)$$

$$\Rightarrow \int_0^4 f(x)\,dx = \left[\frac{1}{2}f(0) + f(1) + f(2) + f(3) + \frac{1}{2}f(4)\right] - \frac{1}{12}\left[f'(4)-f'(0)\right] + \frac{1}{2}\int_0^4 dx\, B_2(x)\, f''(x) \qquad (47)$$

For our purposes, where we intend to stop at 1ST derivatives, we can generalize (47) to

$$\boxed{\int_0^n dx\, f(x) = \frac{1}{2}\left[f(0)+f(n)\right] + \sum_{j=1}^{n-1} f(j) - \frac{1}{12}\left[f'(n) - f'(0)\right] + \frac{1}{2}\int_0^n dx\, B_2(x)\, f''(x)} \tag{48}$$

If we keep expanding the remainder term then we can write

$$\boxed{\begin{aligned}\int_0^n dx\, f(x) = {} & \frac{1}{2}\left[f(0)+f(n)\right] + \sum_{p=1}^{n-1} f(p) - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(n) - f^{(2p-1)}(0)\right] \\ & + \frac{1}{(2q)!}\int_0^n f^{(2q)}(x)\, B_{2q}(x)\, dx \leftarrow \\ & \text{remainder} \equiv R_q\end{aligned}} \equiv R_q \tag{49}$$

We are now interested in the expression for $\int_1^\infty f(x)\,dx$. So we can obtain this result by subtracting the expression in (41) from (48): $[R \equiv \text{remainder}]$

$$\int_1^n dx\, f(x) = \left[\int_0^n - \int_0^1\right] dx\, f(x) = \frac{1}{2}\left[\cancel{f(0)} + f(n)\right] + \sum_{p=1}^{n-1} f(p) - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(n) - \cancel{f^{(2p-1)}(0)}\right] + \cancel{R_q}$$

$$- \frac{1}{2}\left[f(1) + \cancel{f(0)}\right] + \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(1) - \cancel{f^{(2p-1)}(0)}\right] - \cancel{R'_q}$$

OK: But these are not really the same

(50)

$$\therefore \int_1^n dx\, f(x) \cong \frac{1}{2}\left[f(n) - f(1)\right] + \sum_{p=1}^{n-1} f(p) - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(n) - f^{(2p-1)}(1)\right] \tag{51}$$

See also 463.33(135)

$$\therefore \sum_{p=1}^{n-1} f(p) - \int_1^n dx\, f(x) \cong \frac{-1}{2}\left[f(n) - f(1)\right] + \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(n) - f^{(2p-1)}(1)\right] + \underset{(R_q - R'_q)}{\Delta R_q} \tag{52}$$

If we retain only the first derivative contribution, then (52) ⇒

$$\sum_{p=1}^{n-1} f(p) - \int_1^n dx\, f(x) \cong \frac{1}{2}\left[f(1) - f(n)\right] + \frac{1}{12}\left[f'(n) - f'(1)\right] \tag{53}$$

as $n \to \infty$:

$$\boxed{\sum_{p=1}^{\infty} f(p) - \int_1^\infty dx\, f(x) \cong \frac{1}{2}\left[f(1) - f(\infty)\right] + \frac{1}{12}\left[f'(\infty) - f'(1)\right]} \tag{54}$$

EULER-MACLAURIN FORMULA

THE EULER-MASCHERONI CONSTANT γ:

We previously encountered γ when we discussed on pp. 112,113 the following representation for $\Gamma(z)$:

$$\boxed{\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}} \quad (1)$$

$$\gamma = \lim_{N\to\infty} \left(\sum_{1}^{N} \frac{1}{n} - \int_1^N dn \frac{1}{n} \right) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N \right) = 0.57721566\ldots \quad (2)$$

Here we show how to derive the result in (2) using the Euler-Maclaurin formula. Starting from (52) we have ($p \to n$, $n \to N$)

$$\sum_{n=1}^{N-1} f(n) - \int_1^N dn\, f(n) \cong -\frac{1}{2}\left[f(N) - f(1)\right] + \sum_{n=1}^{q} \frac{1}{(2n)!} B_{2n} \left[f^{(2n-1)}(N) - f^{(2n-1)}(1) \right] + \ldots \quad (3)$$

Add & Subtract $f(N) \Rightarrow$

$$\sum_{n=1}^{N} f(n) - \int_1^N dn\, f(n) = -\frac{1}{2}\left[f(N) - f(1)\right] + f(N) + \sum_{n=1} \ldots \quad (4)$$

Hence

$$\boxed{\sum_{n=1}^{N} f(n) - \int_1^N dn\, f(n) \cong \frac{+1}{2}\left[f(N)+f(1)\right] + \sum_{n=1}^{q} \frac{1}{(2n)!} B_{2n} \left[f^{(2n-1)}(N) - f^{(2n-1)}(1) \right] + \ldots} \quad (5)$$

For our purposes: $f(n) = 1/n \Rightarrow f^{(1)}(n) = -1/n^2 \qquad f^{(2)}(n) = +2/n^3$

$$f^{(3)}(n) = -2\cdot3/n^4 \qquad f^{(4)} = 2\cdot3\cdot4/n^5 \;\ldots\; f^{(m)} = (-1)^m m!/n^{m+1} \quad (6)$$

The sum in (5) then evaluates to

$$\Sigma = \frac{1}{2!} B_2 \left[f^{(1)}(N) - f^{(1)}(1) \right] + \frac{1}{4!} B_4 \left[f^{(3)}(N) - f^{(3)}(1) \right] + \frac{1}{6!} \left[f^{(5)}(N) - f^{(5)}(1) \right] + \ldots \quad (7)$$

$$= \frac{1}{2!} B_2 \left[-\frac{1}{N^2} + 1 \right] + \frac{1}{4!} B_4 \left[\frac{-3!}{N^4} + 3! \right] + \frac{1}{6!} B_6 \left[-\frac{5!}{N^6} + 5! \right] + \ldots \quad (8)$$

(γ - continued)

If we now take the limit $N \to \infty$ we find (noting that $f(\infty)=0, f(1)=1$)

$$\boxed{\gamma = \left\{ \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} dn\, f(n) \right\} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) - \int_1^{\infty} \frac{dn}{n} = \frac{1}{2} + \frac{1}{2} B_2 + \frac{1}{4} B_4 + \frac{1}{6} B_6 + \cdots}$$

numerically; $\gamma = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{6}\right) + \frac{1}{4}\left(\frac{-1}{30}\right) + \frac{1}{6}\left(\frac{1}{42}\right)$

$= 0.5 + 0.0833 \quad - 0.0083 \quad + 0.0040 \cong 0.5790$

This compares to the actual value $\gamma = 0.577215\ldots$