SPECIAL TOPICS

PHYS 601

LEAST SQUARES FIT TO A STRAIGHT LINE

Many laws of physics predict a linear correlation between 2 variables,

$$y = ax + b \qquad (1)$$

Examples include OHM's LAW $V = iR$, PLANCK's LAW $E = h\nu$, the photoelectric effect, $E = h\nu - W_0$, etc.

Even when the primitive law is not linear, one often redefines variables or rewrites the law, to make a new relation that is linear. Examples include the KURIE PLOT in β-decay, or the familiar exponential decay law:

$$N = N_0 e^{-\lambda t} \Rightarrow \underbrace{\ln(N/N_0)}_{y} = \underset{a}{(-\lambda)} \underset{x}{t} \qquad (2)$$

This makes testing the law much simpler. To test any such law we imagine that we have carried out an experiment measuring y_i for a series of values x_i; for example, in the photoelectric effect we measure the kinetic energies E_i of emitted electrons as a function of the frequencies ν_i of the incident light and we wish to fit to the theoretical expectation (W_0 = work function)

$$E_i = h\nu_i - W_0 = \mathbf{E_i(\nu_i)} \qquad (3)$$

to find the best value for Planck's constant h and the work function W_0.

The idea behind a least squares fit is this: Let $y_i(\vec{x}; \vec{a}) \equiv y_i(x_i; a, b)$ denote the theoretically expected value of y_i corresponding to x_i, for some assumed values of $a, b \equiv \vec{a}$. In an "ideal world" we can hope to find values of a, b

Such that: $\boxed{y_i\,(\equiv \text{measured value}) = y_i(x_i; \vec{a}) \leftarrow \text{expected value}}$ (4)

The method of least squares is a technique for finding the "best" values of a, b which make $y_i(x_i; \vec{a})$ as close as possible (on average) to y_i. (More generally this is called a MAXIMUM LIKLIHOOD FIT)

$N(E) \approx pE\left[(E_0-E)^2 - \frac{1}{2} m_\nu^2\right]$

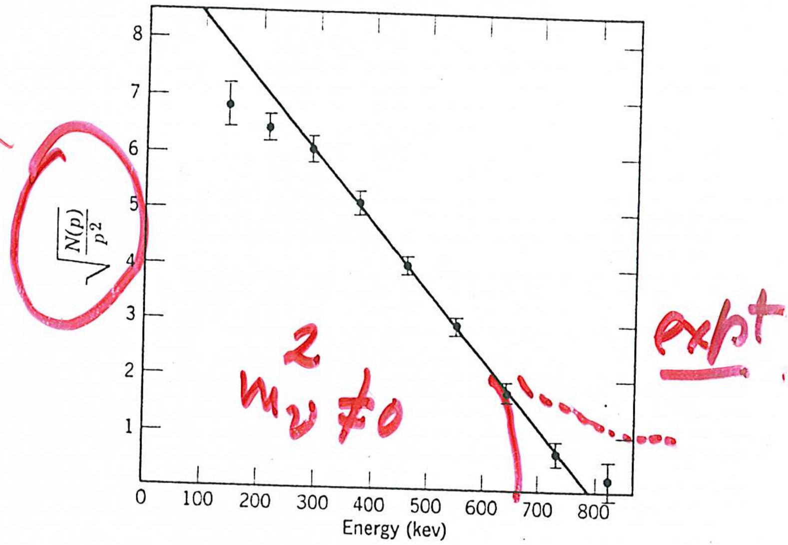

Fig. 2-4 The β spectrum of the neutron plotted as a Kurie plot. $N(p)$ is the number of coincidence per unit momentum interval between β particles and protons resulting from the neutron decay. From (Rob-51).

precision investigations of the allowed β spectra are all in good accord with the allowed shape. It is quite probable that some allowed β spectra of high Z or of exceptionally large ft values may exhibit a slight deviation of a few per cent from the straight Kurie plot at very low energies. These effects theoretically could be attributed to several

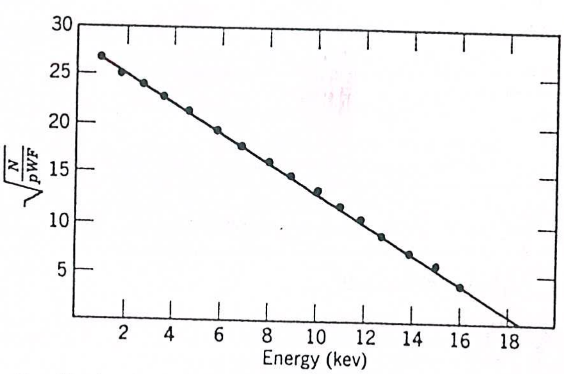

Fig. 2-5 The Kurie plot of H^3 β spectrum obtained by the proportional counter method showing the allowed form to below 1 kev. From (Cu-52).

$\overline{\nu}$ MASS SQUARED (electron based)

Given troubling systematics which result in improbably negative estimators of $m^{2(\mathrm{eff})}_{\nu_e} \equiv \sum_i |U_{ei}|^2\, m^2_{\nu_i}$, in many experiments, we use only KRAUS 05 and LOBASHEV 99 for our average.

VALUE (eV^2)	CL%	DOCUMENT ID		TECN	COMMENT
$-$ 1.1$\pm$ 2.4 OUR AVERAGE					
$-$ 0.6$\pm$ 2.2$\pm$ 2.1		[15] KRAUS	05	SPEC	^{3}H β decay
$-$ 1.9$\pm$ 3.4$\pm$ 2.2		[16] LOBASHEV	99	SPEC	^{3}H β decay
• • • We do not use the following data for averages, fits, limits, etc. • • •					
$-$ 3.7$\pm$ 5.3$\pm$ 2.1		[17] WEINHEIMER	99	SPEC	^{3}H β decay
$-$ 22 $\pm$ 4.8		[18] BELESEV	95	SPEC	^{3}H β decay
129 $\pm$6010		[19] HIDDEMANN	95	SPEC	^{3}H β decay
313 $\pm$5994		[19] HIDDEMANN	95	SPEC	^{3}H β decay
$-$130 $\pm$ 20 $\pm$15	95	[20] STOEFFL	95	SPEC	^{3}H β decay
$-$ 31 $\pm$ 75 $\pm$48		[21] SUN	93	SPEC	^{3}Hβ decay
$-$ 39 $\pm$ 34 $\pm$15		[22] WEINHEIMER	93	SPEC	^{3}H β decay
$-$ 24 $\pm$ 48 $\pm$61		[23] HOLZSCHUH	92B	SPEC	^{3}H β decay
$-$ 65 $\pm$ 85 $\pm$65		[24] KAWAKAMI	91	SPEC	^{3}H β decay
$-$147 $\pm$ 68 $\pm$41		[25] ROBERTSON	91	SPEC	^{3}H β decay

[15] KRAUS 05 is a continuation of the work reported in WEINHEIMER 99. This result represents the final analysis of data taken from 1997 to 2001. Problems with significantly negative squared neutrino masses, observed in some earlier experiments, have been resolved in this work.

[16] LOBASHEV 99 report a new measurement which continues the work reported in BELESEV 95. The data were corrected for electron trapping effects in the source, eliminating the dependence of the fitted neutrino mass on the fit interval. The analysis assuming a pure beta spectrum yields significantly negative fitted $m^2_\nu \approx -(20\text{–}10)$ eV^2. This problem is attributed to a discrete spectral anomaly of about 6×10^{-11} intensity with a time-dependent energy of 5–15 eV below the endpoint. The data analysis accounts for this anomaly by introducing two extra phenomenological fit parameters resulting in a best fit of $m^2_\nu = -1.9 \pm 3.4 \pm 2.2$ eV^2 which is used to derive a neutrino mass limit. However, the introduction of phenomenological fit parameters which are correlated with the derived m^2_ν limit makes unambiguous interpretation of this result difficult.

[17] WEINHEIMER 99 is a continuation of the work reported in WEINHEIMER 93 . Using a lower temperature of the frozen tritium source eliminated the dewetting of the T_2 film, which introduced a dependence of the fitted neutrino mass on the fit interval in the earlier work. An indication for a spectral anomaly reported in LOBASHEV 99 has been seen, but its time dependence does not agree with LOBASHEV 99. Two analyses, which exclude the spectral anomaly either by choice of the analysis interval or by using a particular data set which does not exhibit the anomaly, result in acceptable m^2_ν fits and are used to derive the neutrino mass limit published by the authors. We list the most conservative of the two.

[18] BELESEV 95 (Moscow) use an integral electrostatic spectrometer with adiabatic magnetic collimation and a gaseous tritium sources. This value comes from a fit to a normal Kurie plot above 18300–18350 eV (to avoid a low-energy anomaly), including the effects of an apparent peak 7–15 eV below the endpoint.

[19] HIDDEMANN 95 (Munich) experiment uses atomic tritium embedded in a metal-dioxide lattice. They quote measurements from two data sets.

[20] STOEFFL 95 (LLNL) uses a gaseous source of molecular tritium. An anomalous pileup of events at the endpoint leads to the negative value for m^2_ν. The authors acknowledge that "the negative value for the best fit of m^2_ν has no physical meaning" and discuss possible explanations for this effect.

[21] SUN 93 uses a tritiated hydrocarbon source. See also CHING 95.

[22] WEINHEIMER 93 (Mainz) is a measurement of the endpoint of the tritium β spectrum using an electrostatic spectrometer with a magnetic guiding field. The source is molecular tritium frozen onto an aluminum substrate.

[23] HOLZSCHUH 92B (Zurich) source is a monolayer of tritiated hydrocarbon.

[24] KAWAKAMI 91 (Tokyo) experiment uses tritium-labeled arachidic acid.

[25] ROBERTSON 91 (LANL) experiment uses gaseous molecular tritium. The result is in strong disagreement with the earlier claims by the ITEP group [LUBIMOV 80, BORIS 87 (+ BORIS 88 erratum)] that m_ν lies between 17 and 40 eV. However, the probability of a positive m^2_ν is only 3% if statistical and systematic error are combined in quadrature.

ν MASS (electron based)

These are measurement of m_ν (in contrast to $m_{\overline{\nu}}$, given above). The masses can be different for a Dirac neutrino in the absence of *CPT* invariance. The possible distinction between ν and $\overline{\nu}$ properties is usually ignored elsewhere in these Listings.

VALUE (eV)	CL%	DOCUMENT ID		TECN	COMMENT
<460	68	YASUMI	94	CNTR	^{163}Ho decay
<225	95	SPRINGER	87	CNTR	^{163}Ho decay

ν MASS (muon based)

Limits given below are for the square root of $m^{2(\mathrm{eff})}_{\nu_\mu} \equiv \sum_i |U_{\mu i}|^2\, m^2_{\nu_i}$.

In some of the COSM papers listed below, the authors did not distinguish between weak and mass eigenstates.

OUR EVALUATION is based on OUR AVERAGE for the $\pi^\pm$ mass and the ASSAMAGAN 96 value for the muon momentum for the π^+ decay at rest. The limit is calculated using the unified classical analysis of FELDMAN 98 for a Gaussian distribution near a physical boundary. WARNING: since $m^{2(\mathrm{eff})}_{\nu_\mu}$ is calculated from the differences of large numbers, it and the corresponding limits are extraordinarily sensitive to small changes in the pion mass, the decay muon momentum, and their errors. For example, the limits obtained using JECKELMANN 94, LENZ 98, and the weighted averages are 0.15, 0.29, and 0.19 MeV, respectively.

VALUE (MeV)	CL%	DOCUMENT ID		TECN	COMMENT
<0.19 (CL = 90%) OUR EVALUATION					
<0.17	90	[26] ASSAMAGAN	96	SPEC	$m^2_\nu = -0.016 \pm 0.023$
• • • We do not use the following data for averages, fits, limits, etc. • • •					
<0.15		[27] DOLGOV	95	COSM	Nucleosynthesis
<0.48		[28] ENQVIST	93	COSM	Nucleosynthesis
<0.3		[29] FULLER	91	COSM	Nucleosynthesis
<0.42		[29] LAM	91	COSM	Nucleosynthesis
<0.50	90	[30] ANDERHUB	82	SPEC	$m^2_\nu = -0.14 \pm 0.20$
<0.65	90	CLARK	74	ASPK	$K_{\mu 3}$ decay

[26] ASSAMAGAN 96 measurement of p_μ from $\pi^+ \to \mu^+\nu$ at rest combined with JECKELMANN 94 Solution B pion mass yields $m^2_\nu = -0.016 \pm 0.023$ with corresponding Bayesian limit listed above. If Solution A is used, $m^2_\nu = -0.143 \pm 0.024$ MeV^2. Replaces ASSAMAGAN 94.

[27] DOLGOV 95 removes earlier assumptions (DOLGOV 93) about thermal equilibrium below T_{QCD} for wrong-helicity Dirac neutrinos (ENQVIST 93, FULLER 91) to set more stringent limits.

[28] ENQVIST 93 bases limit on the fact that thermalized wrong-helicity Dirac neutrinos would speed up expansion of early universe, thus reducing the primordial abundance. FULLER 91 exploits the same mechanism but in the older calculation obtains a larger production rate for these states, and hence a lower limit. Neutrino lifetime assumed to exceed nucleosynthesis time, $\sim$ 1 s.

[29] Assumes neutrino lifetime >1 s. For Dirac neutrinos only. See also ENQVIST 93.

[30] ANDERHUB 82 kinematics is insensitive to the pion mass.

ν MASS (tau based)

The limits given below are the square roots of limits for $m^{2(\mathrm{eff})}_{\nu_\tau} \equiv \sum_i |U_{\tau i}|^2\, m^2_{\nu_i}$.

In some of the ASTR and COSM papers listed below, the authors did not distinguish between weak and mass eigenstates.

VALUE (MeV)	CL%	EVTS	DOCUMENT ID		TECN	COMMENT
< 18.2	95		[31] BARATE	98F	ALEP	1991–1995 LEP runs
• • • We do not use the following data for averages, fits, limits, etc. • • •						
< 28	95		[32] ATHANAS	00	CLEO	E^{ee}_{cm}= 10.6 GeV
< 27.6	95		[33] ACKERSTAFF	98T	OPAL	1990–1995 LEP runs
< 30	95	473	[34] AMMAR	98	CLEO	E^{ee}_{cm} = 10.6 GeV
< 60	95		[35] ANASTASSOV	97	CLEO	E^{ee}_{cm}= 10.6 GeV
< 0.37 or >22			[36] FIELDS	97	COSM	Nucleosynthesis
< 68	95		[37] SWAIN	97	THEO	m_τ, τ_τ, τ partial widths
< 29.9	95		[38] ALEXANDER	96M	OPAL	1990–1994 LEP runs
<149			[39] BOTTINO	96	THEO	π, μ, τ leptonic decays
<1 or >25			[40] HANNESTAD	96C	COSM	Nucleosynthesis
< 71	95		[41] SOBIE	96	THEO	m_τ, τ_τ, B($\tau^- \to e^-\overline{\nu}_e\nu_\tau$)
< 24	95	25	[42] BUSKULIC	95H	ALEP	1991–1993 LEP runs
< 0.19			[43] DOLGOV	95	COSM	Nucleosynthesis
< 3			[44] SIGL	95	ASTR	SN 1987A
< 0.4 or > 30			[45] DODELSON	94	COSM	Nucleosynthesis
< 0.1 or > 50			[46] KAWASAKI	94	COSM	Nucleosynthesis
155–225			[47] PERES	94	THEO	π,K,μ,τ weak decays
< 32.6	95	113	[48] CINABRO	93	CLEO	$E^{ee}_{\mathrm{cm}} \approx$ 10.6 GeV
< 0.3 or > 35			[49] DOLGOV	93	COSM	Nucleosynthesis
< 0.74			[50] ENQVIST	93	COSM	Nucleosynthesis
< 31	95	19	[51] ALBRECHT	92M	ARG	E^{ee}_{cm}= 9.4–10.6 GeV
< 0.3			[52] FULLER	91	COSM	Nucleosynthesis
< 0.5 or > 25			[53] KOLB	91	COSM	Nucleosynthesis
< 0.42			[52] LAM	91	COSM	Nucleosynthesis

[31] BARATE 98F result based on kinematics of 2939 $\tau^- \to 2\pi^-\pi^+\nu_\tau$ and 52 $\tau^- \to 3\pi^-2\pi^+(\pi^0)\nu_\tau$ decays. If possible 2.5% excited a_1 decay is included in 3-prong sample analysis, limit increases to 19.2 MeV.

[32] ATHANAS 00 bound comes from analysis of $\tau^- \to \pi^-\pi^+\pi^-\pi^0\nu_\tau$ decays.

[33] ACKERSTAFF 98T use $\tau \to 5\pi^\pm\nu_\tau$ decays to obtain a limit of 43.2 MeV (95%CL). They combine this with ALEXANDER 96M value using $\tau \to 3h^\pm\nu_\tau$ decays to obtain quoted limit.

[34] AMMAR 98 limit comes from analysis of $\tau^- \to 3\pi^-2\pi^+\nu_\tau$ and $\tau^- \to 2\pi^-\pi^+2\pi^0\nu_\tau$ decay modes.

Method: Suppose that $i = 1, \ldots, N$ denotes the N measurements of y_i made at N values of x_i. We seek the values of $\vec{a} = (a, b)$ which minimize the function

$$\boxed{\chi^2 = \sum_{i=1}^{N} \frac{[y_i - y(x_i; \vec{a})]^2}{\sigma_i^2}} \tag{5}$$

Here we imagine that corresponding to each x_i there is a measurement $(y_i \pm \sigma_i)$.

Specifically (5) can be written as

"expected"

"measured"

"chi-square"

$$\boxed{\chi^2 = \sum_{i=1}^{N} \frac{[y_i - (ax_i + b)]^2}{\sigma_i^2}} \tag{6}$$

In (6), x_i and $y_i \pm \sigma_i$ are given inputs: We are then seeking to find the values of a, b which minimize χ^2. Evidently this serves to minimize the difference between the measured and expected (predicted) values of y_i. Note that this gives a global fit for a, b; the best values on average; to this end we observe that the contribution to χ^2 from each pair $(x_i, y_i \pm \sigma_i)$ is weighted by $1/\sigma_i^2$ so that better-determine points have more influence on χ^2.

The condition that we minimize χ^2 then gives 2 conditions which fix a & b:

$$0 = \frac{\partial \chi^2}{\partial a} = \frac{\partial}{\partial a} \sum_{i=1}^{N} \frac{[y_i - (ax_i + b)]^2}{\sigma_i^2} = \sum_{i=1}^{N} \frac{(-2x_i)[y_i - (ax_i + b)]}{\sigma_i^2} \tag{7}$$

$$0 = \frac{\partial \chi^2}{\partial b} = \sum_{i=1}^{N} \frac{(-2)[y_i - (ax_i + b)]}{\sigma_i^2} \tag{8}$$

Dropping the irrelevant overall factors of 2 we have

$$\frac{\partial \chi^2}{\partial a} = 0 \Rightarrow \sum_{i=1}^{N} \left\{ \frac{x_i y_i}{\sigma_i^2} - \frac{a x_i^2}{\sigma_i^2} - \frac{b x_i}{\sigma_i^2} \right\} = 0 \tag{9}$$

$$\frac{\partial \chi^2}{\partial b} = 0 \Rightarrow \sum_{i=1}^{N} \left\{ \frac{y_i}{\sigma_i^2} - \frac{a x_i}{\sigma_i^2} - b \frac{1}{\sigma_i^2} \right\} = 0 \tag{10}$$

These can be written in a more useful form as:

$$a\left(\sum_{i=1}^{N} \frac{x_i^2}{\sigma_i^2}\right) + b\left(\sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}\right) = \sum_{i=1}^{N} \frac{x_i y_i}{\sigma_i^2} \tag{11}$$

$$a\left(\sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}\right) + b\left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\right) = \sum_{i=1}^{N} \frac{y_i}{\sigma_i^2} \tag{12}$$

These can be written in the form of a matrix equation as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \equiv \underbrace{\begin{pmatrix} \sum x_i^2/\sigma_i^2 & \sum x_i/\sigma_i^2 \\ \sum x_i/\sigma_i^2 & \sum 1/\sigma_i^2 \end{pmatrix}}_{g}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum x_i y_i/\sigma_i^2 \\ \sum y_i/\sigma_i^2 \end{pmatrix} \tag{13}$$

If we define a VARIANCE MATRIX $V = g^{-1}$ then evidently,

$$g^{-1} g \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = g^{-1}\begin{pmatrix} \cdots \\ \cdots \end{pmatrix} = V\begin{pmatrix} \cdots \\ \cdots \end{pmatrix} \tag{14}$$

Hence finally:

$$\boxed{\begin{pmatrix} a \\ b \end{pmatrix} = V\begin{pmatrix} \sum_i x_i y_i/\sigma_i^2 \\ \sum_i y_i/\sigma_i^2 \end{pmatrix}} \tag{15}$$

To compute $V = g^{-1}$:

$$g_{ij} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad (g^{-1})_{ij} = \frac{(\mathrm{Adj}\, g)_{ij}}{\det g} \tag{16}$$

MEMORIZE!!

$$\det g = (a_{11} a_{22} - a_{12} a_{21})$$

$$(\mathrm{Adj}\, g)_{11} = (-1)^{1+1} a_{22} \qquad (\mathrm{Adj}\, g)_{12} = (-1)^{1+2} a_{12} \tag{17}$$

$$(\mathrm{Adj}\, g)_{22} = (-1)^{2+2} a_{11} \qquad (\mathrm{Adj}\, g)_{21} = (-1)^{2+1} a_{21}$$

Hence

$$\boxed{V = g^{-1} = \frac{1}{\det g}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}} \tag{18}$$

It can be shown that V has the form

$$V = \begin{pmatrix} \sigma_a^2 & \rho_{ab}\sigma_a\sigma_b \\ \rho_{ab}\sigma_a\sigma_b & \sigma_b^2 \end{pmatrix} \qquad (19)$$

where σ_a and σ_b represent the errors in the determinations of a and b respectively, and ρ_{ab} is the correlation coefficient between a and b. Combining (18) & (19) we find

$$\boxed{\sigma_a^2 = \frac{a_{22}}{\det g} = \frac{\sum_i 1/\sigma_i^2}{\det g} \; ; \; \sigma_b^2 = \frac{a_{11}}{\det g} = \frac{\sum_i x_i^2/\sigma_i^2}{\det g} \qquad (20)}$$

$$\boxed{\det g = \left(\sum_i x_i^2/\sigma_i^2\right)\left(\sum_j 1/\sigma_j^2\right) - \left(\sum_i x_i/\sigma_i^2\right)^2 \qquad (21)}$$

To determine the actual numerical values of a & b we combine Eqs. (15) & (18):

$$\begin{pmatrix} a \\ b \end{pmatrix} = V \begin{pmatrix} \sum_i x_i y_i/\sigma_i^2 \\ \sum_i y_i/\sigma_i^3 \end{pmatrix} = \frac{1}{\det g}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} \sum_i x_i y_i/\sigma_i^2 \\ \sum_i y_i/\sigma_i^2 \end{pmatrix} \qquad (22)$$

Then:

$$\boxed{a = \frac{1}{\det g}\left\{a_{22}\sum_i \frac{x_i y_i}{\sigma_i^2} - a_{12}\sum_i \frac{y_i}{\sigma_i^2}\right\} = \frac{1}{\det g}\left\{\left(\sum_j \frac{1}{\sigma_j^2}\right)\left(\sum_i \frac{x_i y_i}{\sigma_i^2}\right) - \left(\sum_j \frac{x_j}{\sigma_j^2}\right)\left(\sum_i \frac{y_i}{\sigma_i^2}\right)\right\} \qquad (23)}$$

$$\boxed{b = \frac{1}{\det g}\left\{-a_{21}\sum_i \frac{x_i y_i}{\sigma_i^2} + a_{11}\sum_i \frac{y_i}{\sigma_i^2}\right\} = \frac{1}{\det g}\left\{-\left(\sum_j \frac{x_j}{\sigma_j^2}\right)\left(\sum_i \frac{x_i y_i}{\sigma_i^2}\right) + \left(\sum_j \frac{x_j^2}{\sigma_j^2}\right)\left(\sum_i \frac{y_i}{\sigma_i^2}\right)\right\} \qquad (24)}$$

<u>Example:</u> Consider the following data set:

i	x_i	y_i	σ_i	σ_i^2
1	1	1	1	1
2	2	3	2	4
3	3	2	1	1

In the notation of the PARTICLE DATA GROUP (PDG) the quantities that we need are:

$$\boxed{S_1, S_x, S_y, S_{xx}, S_{xy} = \sum_{i=1}^{5} (1, x_i, y_i, x_i^2, x_i y_i)/\sigma_i^2} \quad (25)$$

In terms of these quantities we have

$$\boxed{\begin{aligned} \det g &= S_{xx}S_1 - S_x^2 \; ; \; a = \frac{1}{\det g}\left(S_1 S_{xy} - S_x S_y\right) \; ; \; \sigma_a^2 = \frac{S_1}{\det g} \\ b &= \frac{1}{\det g}\left(-S_x S_{xy} + S_{xx} S_y\right) ; \; \sigma_b^2 = \frac{S_{xx}}{\det g} \end{aligned}} \quad (26)$$

Numerically: $S_1 = \frac{1}{1} + \frac{1}{4} + \frac{1}{1} = 2.25$; $S_x = \frac{1}{1} + \frac{2}{4} + \frac{3}{1} = 4.5$

$S_y = \frac{1}{1} + \frac{3}{4} + \frac{2}{1} = 3.75$; $S_{xx} = \frac{1}{1} + \frac{4}{4} + \frac{9}{1} = 11.0$

$S_{xy} = \frac{1}{1} + \frac{6}{4} + \frac{6}{1} = 8.5$

$\det g = (11.0)(2.25) - (4.5)^2 = 4.5$

$a = \frac{1}{4.5}\left[(2.25)(8.5) - (4.5)(3.75)\right] = 0.5$; $\sigma_a = \pm 0.7071$

$a = 0.5 \pm 0.7071$

$b = \frac{1}{4.5}\left[(-4.5)(8.5) + (11.0)(3.75)\right] = 0.667$; $\sigma_b = \pm 1.563$

Calculation of χ^2 for the Fit:

Returning to Eq. (6), we have solved for a, b by minimizing χ^2:

$$\chi^2 = \sum_{i=1}^{N} \frac{[y_i - (ax_i + b)]^2}{\sigma_i^2} \tag{1}$$

measured — expected

The procedure that we have followed produces the "best" values of a, b to minimize χ^2. However, in the end the question is "how well do the data support the straight line hypothesis?" In an "idealized" world we might expect a perfect agreement $\Rightarrow \chi^2 \equiv 0$. However, for real data this never happens, and hence χ^2 is a measure of the overall quality of the agreement between theory and experiment. Expanding (1) we have:

$$\chi^2 = \sum_{i=1}^{N} \frac{[y_i^2 + a^2x_i^2 + b^2 - 2a\,x_iy_i - 2by_i + 2ab\,x_i]}{\sigma_i^2} \tag{2}$$

If we introduce the additional sum [see 247(25)]: $S_{yy} \equiv \sum_{i=1}^{N} \frac{y_i^2}{\sigma_i^2}$ (3)

Then

$$\boxed{\chi^2 = S_{yy} + a^2 S_{xx} + b^2 S_1 - 2a S_{xy} - 2b S_y + 2ab S_x} \tag{4}$$

What is a good fit? Suppose we are fitting N data points to any curve which is characterized by p parameters (here p = 2 denoting a and b). We define the number of degrees of freedom d as

$$\boxed{d = \underset{(\text{d.o.f.})}{\text{degrees of freedom}} \equiv N - p} \tag{5}$$

The measure of the quality of the fit is then

$$\boxed{\chi^2/d = \chi^2/\text{d.o.f}} \quad (6)$$

Tables give Confidence Levels a a function of χ^2/d, but as a rougher "rule of thumb" a good fit corresponds to $\chi^2/d \simeq 1$.

To understand this in more detail we note that the contribution to χ^2 from each point (x_i, y_i) is the square of the distance of each y_i from the line measured in units of σ_i.

Evidently, the more points one fits the line to, the larger χ^2 will be, since each contribution to χ^2 is a positive definite term. Hence even if the fit were very good in the sense that all the points fell near the line, χ^2 would be large (and the fit might not seem that good). For this reason, the relevant parameter is $\chi^2/\text{d.o.f}$ in which the growth of χ^2 in the numerator is offset by the growth of d.o.f. in the denominator. Note also that in dividing the value of χ^2 by $d = \text{d.o.f.}$, and not simply by the number of points, we avoid the embarrassing situation of fitting a straight line to 2 points: With the definition as above d.o.f $= 0$ so that the expression in (6) $\to 0/0$ which is meaningless, as it should be for a straight line fit to 2 points.

The homework problem will serve to clarify some of these points.

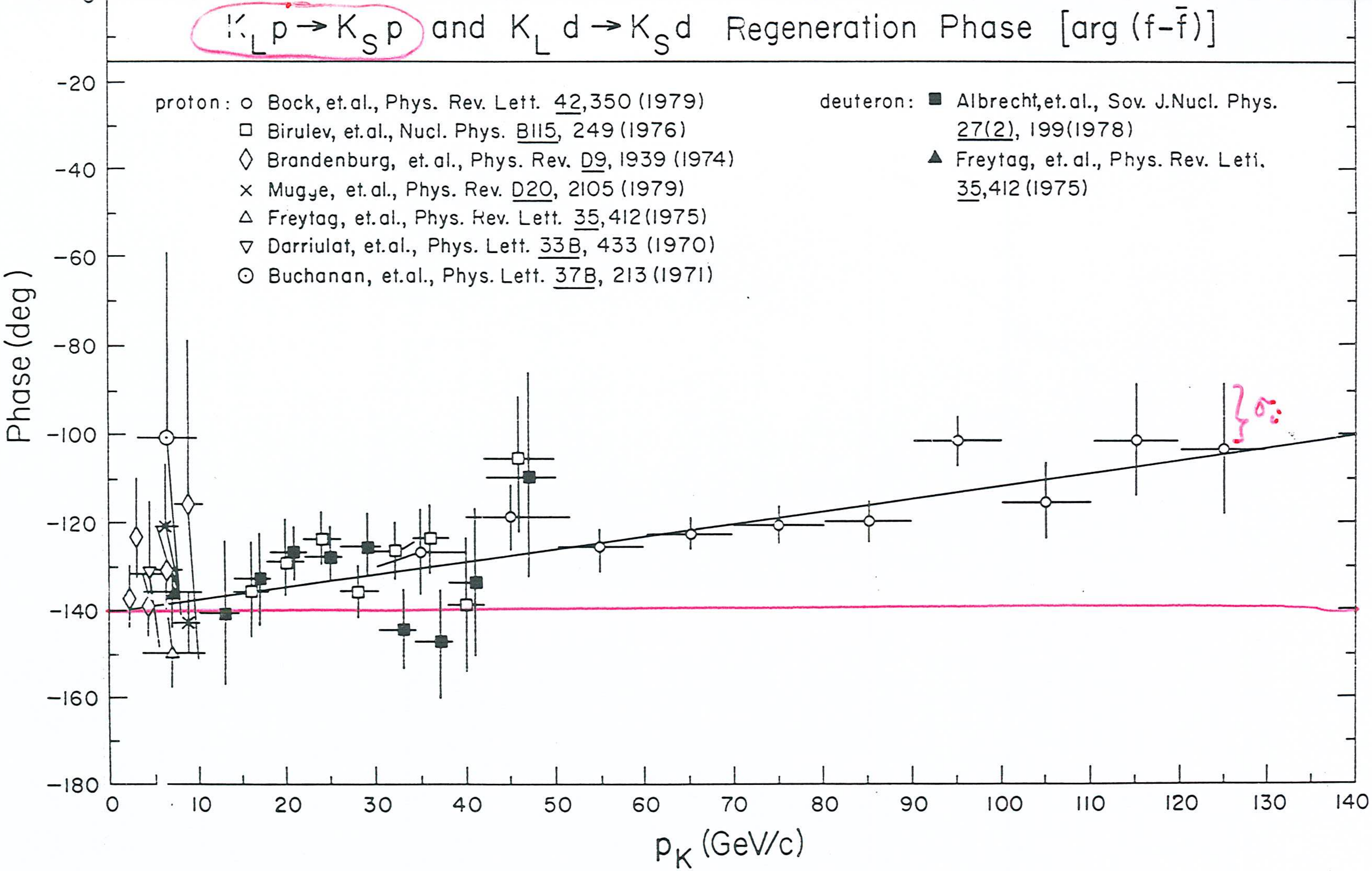

Fig. 1. Plot of the regeneration phase in hydrogen vs. kaon momentum. The straight line is a fit to the data points of Bock et al. [5]. Essentially all plotted values of the phase are derived from interference measurements under the assumption that ϕ_{+-} = const. $\cong 45°$ as described in the text. See ref. [3] for details.

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Reanalysis of the Eötvös Experiment

Ephraim Fischbach[(a)]
Institute for Nuclear Theory, Department of Physics, University of Washington, Seattle, Washington 98195

Daniel Sudarsky, Aaron Szafer, and Carrick Talmadge
Physics Department, Purdue University, West Lafayette, Indiana 47907

and

S. H. Aronson
Physics Department, Brookhaven National Laboratory, Upton, New York 11973
(Received 7 November 1985)

We have carefully reexamined the results of the experiment of Eötvös, Pekár, and Fekete, which compared the accelerations of various materials to the Earth. We find that the Eötvös-Pekár-Fekete data are sensitive to the composition of the materials used, and that their results support the existence of an intermediate-range coupling to baryon number or hypercharge.

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Recent geophysical determinations of the Newtonian constant of gravitation G have reported values which are consistently higher than the laboratory value G_0.[1] With the assumption that the discrepancy between these two sets of values is a real effect, one interpretation of these results is that they are the manifestation of a non-Newtonian coupling of the form

$$V(r) = -G_\infty \frac{m_1 m_2}{r}(1+\alpha e^{-r/\lambda}) = V_N(r) + \Delta V(r). \quad (1)$$

Here $V_N(r)$ is the usual Newtonian potential energy for two masses $m_{1,2}$ separated by a distance r, and G_∞ is the Newtonian constant of gravitation for $r \to \infty$. The geophysical data can then be accounted for quantitatively if α and λ have the values[2]

$$\alpha = -(7.2 \pm 3.6)\times 10^{-3}, \quad \lambda = 200 \pm 50 \text{ m}. \quad (2)$$

If $\Delta V(r)$ actually describes the effects of a new force, and is not just a parametrization of some other systematic effects, then its presence would be expected to manifest itself elsewhere as well. Recently, we have undertaken an exhaustive search for the presence of such a force in other systems. Our analysis, to be presented elsewhere,[3] leads to the conclusion that if such a force existed it would show up at present sensitivity levels in only three additional places: (i) the K^0-$\bar{K}^0$ system at high laboratory energies, where in fact anomalous effects have previously been reported[4]; (ii) a comparison of satellite and terrestrial determinations[5] of the local gravitational acceleration **g**; and (iii) the original Eötvös experiment[6] which compared the acceleration of various materials to the Earth. We note that the subsequent repetitions of the Eötvös experiment by Roll, Krotkov, and Dicke[7] and by Braginskii and Panov[8] compared the gravitational accelerations of a pair of test materials to the Sun, and hence would not have been sensitive to the intermediate-range force described by Eqs. (1) and (2). Motivated by our general analysis, we returned to the Eötvös experiment and asked whether there is evidence in their data of the presence of $\Delta V(r)$ in Eq. (1). Although the Eötvös experiment has been universally interpreted as having given null results, we find in fact that this is not the case. Furthermore, we will demonstrate explicitly that the published data of Eötvös, Pekár and

Fekete[6] (EPF) not only suggest the presence of a non-Newtonian coupling $\Delta V(r)$, but also strongly support the specific values of the parameters α and λ in Eq. (2), which emerge from an analysis of the geophysical data.

Guided by the observations that (a) $\alpha < 0$, which indicates a *repulsive* force, and (b) anomalous effects have been reported in the K^0-$\bar{K}^0$ system as well, we consider the effects of a massive hypercharge field whose quanta (hyperphotons) have a mass $m_Y = \lambda^{-1} = 1\times10^{-9}$ eV. The exchange of a hyperphoton then gives rise to a potential having the same form as $\Delta V(r)$ in Eq. (1), with α being related to the unit of hypercharge f by

$$f^2/G_0 m_p^2 \cong -\alpha/(1+\alpha), \tag{3}$$

where m_p is the proton mass. Consider the relative accelerations of two objects 1 and 2 with masses $m_{1,2}$ and hypercharges (or baryon numbers) $B_{1,2}$. Because of the presence of $\Delta V(r)$ the accelerations $a_{1,2}$ of these objects to the Earth will no longer be the universal Newtonian value g, but will differ by an amount $\Delta a = a_1 - a_2$ given by

$$\frac{\Delta a}{g} = \frac{f^2 \epsilon(R/\lambda)}{G_0 m_{\mathrm{H}}^2}\left(\frac{B_\oplus}{\mu_\oplus}\right)\left(\frac{B_1}{\mu_1} - \frac{B_2}{\mu_2}\right). \tag{4}$$

Here μ_i denotes the mass m_i in units of atomic hydrogen, with $m_{\mathrm{H}} = m({}_1\mathrm{H}^1) = 1.007\,825\,19(8)$ u, and we can take $B_\oplus/\mu_\oplus \cong 1$ for present purposes. $\epsilon(R/\lambda)$ arises from integration of the intermediate-range hypercharge distribution over the Earth, assumed to be a uniform sphere of radius R, and is given by $(x = R/\lambda)$

$$\epsilon(x) = \frac{3(1+x)}{x^3} e^{-x}(x\cosh x - \sinh x). \tag{5}$$

For $\lambda \to \infty$, $\epsilon(0) \to 1$, and (4) reduces to the result of Lee and Yang.[9] However, the limit of interest to us here is $x >> 1$ in which case $\epsilon(x) \cong 3/2x$.

Equation (4) can now be compared directly to the results of EPF, where in their notation $\Delta a/g = \kappa_1 - \kappa_2 = \Delta\kappa$. Table I gives $\Delta\kappa$ for each of the nine pairs of materials measured by EPF, exactly as their result is quoted on the indicated page of Ref. 6. For each of the pairs in which the composition of both samples can be established (see discussion below), we also tabulate $\Delta(B/\mu) = B_1/\mu_1 - B_2/\mu_2$ using the data of Ref. 10. In the computation of B/μ for each material, care has been taken to average over all the isotopes of each element, and to weight the contribution of each element in a compound according to the appropriate chemical formula. Among the substances appearing in Table I, Cu, Pt, and water require no further description, crystalline copper sulfate has the formula $CuSO_4 \cdot 5H_2O$, and the $CuSO_4$ solution consisted of 20.61 g of crystalline copper sulfate in 49.07 g of water. By contrast, magnalium is an aluminum-magnesium alloy of varying composition, with typical Al:Mg ratios being in the range 95:5–70:30. Although the exact composition of the magnalium alloy used by EPF is not given, B/μ for Al and Mg are very nearly equal so that B/μ for any magnalium alloy would fall in the narrow range

$$1.008\,45 \text{ (pure Mg)} \lesssim B/\mu(\text{magnalium}) \lesssim 1.008\,51 \text{ (pure Al)}. \tag{6}$$

The results in Table I assume a composition Al:Mg = 90:10, which is one of the more common alloys. The remaining material whose composition can be established with some certainty is asbestos, since 95% of asbestos production is a fibrous form of the mineral serpentine called chrysotile,[11] whose chemical formula is $Mg_3Si_2O_5(OH)_4$. In addition to measuring the relative acceleration of various pairs of materials, EPF also compared the accelerations of the reactants before and after the chemical reaction

$$Ag_2SO_4 + 2FeSO_4 \to 2Ag + Fe_2(SO_4)_3. \tag{7}$$

TABLE I. Summary of EPF results for $\Delta\kappa$, and page quoted from Ref. 6, along with the computed values of $\Delta(B/\mu)$. Ag-Fe-SO_4 refers to the reactants before and after the chemical reaction described by Eq. (7).

Materials compared	Page quoted	$10^8\Delta\kappa$	$10^3\Delta(B/\mu)$
Cu-Pt	37	$+0.4 \pm 0.2$	+0.94
Magnalium-Pt	34	$+0.4 \pm 0.1$	+0.50
Ag-Fe-SO_4	39	0.0 ± 0.2	0.00
Asbestos-Cu	47	-0.3 ± 0.2	−0.74
$CuSo_4 \cdot 5H_2O$-Cu	44	-0.5 ± 0.2	−0.86
$CuSO_4$(solution)-Cu	45	-0.7 ± 0.2	−1.42
Water-Cu	42	-1.0 ± 0.2	−1.71
Snakewood-Pt	35	-0.1 ± 0.2	?
Tallow-Cu	48	-0.6 ± 0.2	?

Since B/μ is the same before and after the reaction, $\Delta\kappa$ should be zero in this case, which is indeed what EPF found. The remaining materials used by EPF are *schlangenholz* (snakewood) and *talg* (tallow, grease, suet, etc.) whose exact compositions cannot be established. In particular, the amount of water in each of these is unknown, and since water has a relatively low value of B/μ, the effective value of B/μ for the sampel could vary over a wide range depending on its water content.

In Fig. 1 we plot the measured value of $\Delta\kappa$ versus the computed values of $\Delta(B/\mu)$ using the data given in Table I. We see immediately that the EPF data clearly exhibit the linear relationship between $\Delta\kappa$ and $\Delta(B/\mu)$ expected from Eq. (4). Furthermore, the solid line resulting from a least-squares fit to the data passes through the origin, as it should if Eq. (4) holds. Finally, the slope of the line is in remarkably good agreement with the value expected from the parameters in Eq. (2) which arise from the geophysical data. Specifically, we find from the least-squares fit that the equation of the line is

$$\begin{aligned} &\Delta\kappa = a\,\Delta(B/\mu) + b,\\ &a = (5.65 \pm 0.71)\times 10^{-6},\\ &b = (4.83 \pm 6.44)\times 10^{-10},\\ &\chi^2 = 2.1 \text{ (5 degrees of freedom).} \end{aligned} \tag{8}$$

Combining (4) and (8), we can solve for $f^2\epsilon(R/\lambda)$,

$$[f^2\epsilon(R/\lambda)]_{\text{Eötvös}} = G_0 m_{\mathrm{H}}^2 a = (4.6 \pm 0.6)\times 10^{-42} e^2, \tag{9}$$

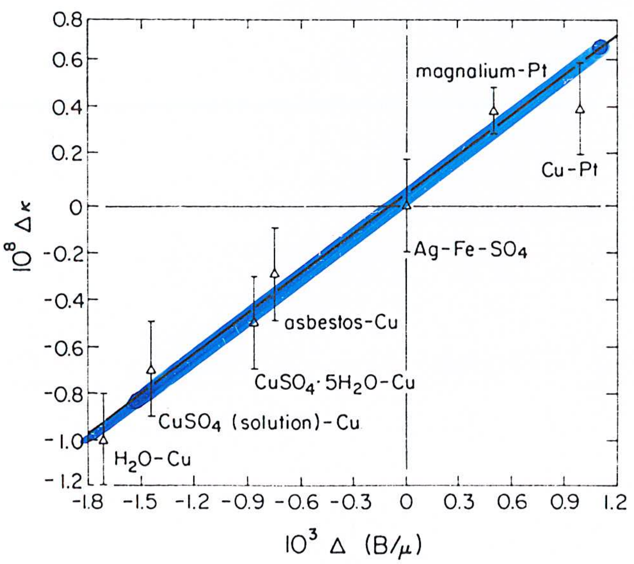

FIG. 1. Plot of $\Delta\kappa$ vs $\Delta(B/\mu)$ using the data in Table I. Ag-Fe-SO_4 refers to the reactants before and after the chemical reaction described by Eq. (7). The solid line represents the results of a least-squares fit to the data.

where e is the electric charge in Gaussian units. This should be compared to the value derived from the geophysical data in Eq. (2),

$$[f^2\epsilon(R/\lambda)]_{\text{geophysical}} = (2.8 \pm 1.5)\times 10^{-43} e^2. \tag{10}$$

The agreement between these two results is surprisingly good, particularly in view of the simple model of the Earth that has been used in deriving (4) and (9). If λ is in fact on the order of 200 m, then the details of the local matter distribution will clearly modify the functional form of $\epsilon(R/\lambda)$, and could lead to improved agreement between (9) and (10). If the potential in Eqs. (1) and (2) describes a coupling to hypercharge, as we have assumed, then it should also give rise to an anomalous energy dependence of the fundamental K^0-$\bar{K}^0$ parameters such as the K_L-K_S mass difference Δm, the K_S lifetime τ_S, and the CP nonconserving parameter η_{+-}. Here the intermediate-range nature of the coupling is crucial in understanding the effects that arise. As we discuss in Ref. 3, the specific values of α and λ in (2), which account for both the geophysical data and the Eötvös results, may also explain the kaon data as well, both qualitatively and quantitatively.

The possibility that the three effects that we have discussed do in fact have a common origin can be directly tested in several ways. To start with, the Eötvös experiment itself should be repeated with greater sensitivity, and with a variety of materials whose precise composition is known. As has been noted elsewhere,[12] the composition dependence of the Eötvös anomaly[13] $\Delta a/g$ can be used to rule out various possible explanations of this effect. In particular, we show in Ref. 3 that neither a coupling to lepton number nor a recently proposed model of Lorentz noninvariance can account for the data of Ref. 6. While a repeat of the Eötvös experiment with better sensitivity may be possible with modern techniques, it may be more practical simply to compare the times of flight of different test masses dropped from the same height, in an updated version of the Galileo experiment.[14] To achieve a sensitivity sufficient for our purposes, say $\Delta a/g = 10^{-10}$, would require measurement of the time of flight to within 0.1 ns over a distance of 10 m which is within the realm of feasibility. In addition, one can attempt to improve the measurement of Δg, the difference between the locally measured value of $\mathbf{g}$ and that implied by satellite data. Evidently satellite measurements would not be sensitive to $\Delta V(r)$ in (1) and (2), whereas local measurements would, and it follows from (1) and (2) that $\Delta g/g$ should be approximately 2×10^{-7}. An analysis of the available data by Rapp[15] gives a value $\Delta g/g \approx (6 \pm 10)\times 10^{-7}$, but the prospects for improving on this result are somewhat uncertain. Finally, if we take seriously the existence of a hypercharge field, then one can search directly for hyperphotons γ_Y via their cosmological effects, and in decays such as $K^0 \to 2\pi + \gamma_Y$. Following Weinberg,[16] we note that the branching ratio for this mode is

$$\frac{\Gamma(K^0 \to 2\pi + \gamma_Y)}{\Gamma(K^0 \to 2\pi)} = \frac{f^2}{8\pi^2}\frac{E_{\max}^2}{m_Y^2}, \qquad (11)$$

where $E_{\max} \ll m_K$ is the maximum hyperphoton energy detected. For f and m_Y as given in (2) and (3), and $E_{\max} = 100$ MeV, the branching ratio is 6×10^{-9}. This is safely below the level where hyperphotons could have been detected in the course of other experiments, but at the same time is large enough so that a direct search for this mode may prove possible. From a cosmological point of view, hyperphotons would act as a massive but very weakly interacting constituent of interstellar space, and could thus help account for the missing mass of the Universe.

We are indebted to Frank Stacey for communicating the results in Eq. (2) prior to publication, and to Peter Buck for translating parts of Ref. 6. We also wish to thank Mark Haugan, Wick Haxton, Ernest Henley, Fred Raab, and Richard Rapp for helpful conversations. One of us (E.F.) wishes to thank the Institute for Nuclear Theory at the University of Washington for its hospitality during the course of the research. This work was supported in part by the U. S. Department of Energy.

Note added.—R. H. Dicke (private communication) has raised with us the question of whether some systematic effect in the EPF experiment could simulate the observed correlation between $\Delta\kappa$ and $\Delta(B/\mu)$. He proposed an interesting model in which thermal gradients could lead to a correlation between $\Delta\kappa$ and the quantity $(a + b/\rho_1 - c/\rho_2)$, where $\rho_{1,2}$ are the densities of the samples and a, b, and c free parameters. We have investigated this model, and others involving $\rho_{1,2}$, and have found that none of these show a correlation with $\Delta\kappa$. These results will be presented in detail in Ref. 3, where we will also show that they are a consequence of two special properties of B/μ: (1) it has an anomalously low value for hydrogen, and (2) it has a maximum near Fe and is lower toward either end of the Periodic Table. We wish to thank Professor Dicke for stimulating us to investigate this question.

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[13]It is interesting to note that on p. 65 of Ref. 6 EPF summarize their data as if all the indicated samples were actually measured against a Pt standard, notwithstanding the fact that in most of the measurements the actual standard was Cu. The effect of combining, say, $\Delta\kappa(H_2O\text{-}Cu)$ and $\Delta\kappa(Cu\text{-}Pt)$ to infer $\Delta\kappa(H_2O\text{-}Pt)$ is to reduce the magnitude of the observed nonzero effect from 5σ to 2σ. Any suggestion of a nonzero effect was further reduced by choosing Pt rather than Cu as the standard since, had the opposite choice been made, the signs of all the nonzero $\Delta\kappa$ would have been the same, and might thus have pointed to a possible systematic effect.

[14]Improvements in the Eötvös and Galileo experiments will be the subject of a separate paper.

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"THE RULE OF 72"

This is a simple mnemonic to calculate how long it will take to double how much money you have, given an assumed constant rate of growth r (% per year).

At $t=0$ you have \$d. After 1 year you will then have \$d(1+r). After n years you will then have \$d$(1+r)^n$.

Hence we want to find n such that

$$\$d(1+r)^n = 2\$d \Rightarrow (1+r)^n = 2 \Rightarrow \boxed{n \ln(1+r) = \ln 2} \quad (1)$$

If we assume a conventionally modest value of $r \ll 1$, we can then expand,

$$\ln(1+r) \cong r \Rightarrow n \ln(1+r) \cong nr = \ln 2 \cong 0.693 \quad (2)$$

Hence

$$n = \frac{0.693}{r} = \frac{69.3}{100r} \cong \frac{72}{R} \qquad R = \text{growth rate in } 100 \times \% \text{ } (6\% \to 6) \quad (3)$$

Final result: $\boxed{n \cong \frac{72}{R} = \text{"rule of 72"}}$ (4)

Examples:	Rule (4)	Exact (1)
$r = 4\% \Rightarrow R = 4$	$n = 18$	17.7
$r = 6\% \Rightarrow R = 6$	$n = 12$	11.9
$r = 8\% \Rightarrow R = 8$	$n = 9$	9.006
$r = 9\% \Rightarrow R = 9$	$n = 8$	8.04
$r = 12\% \Rightarrow R = 12$	$n = 6$	6.11