Wilson Loops in Large N Field Theories

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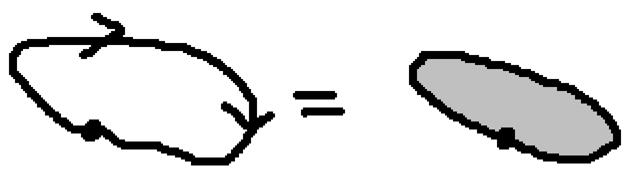
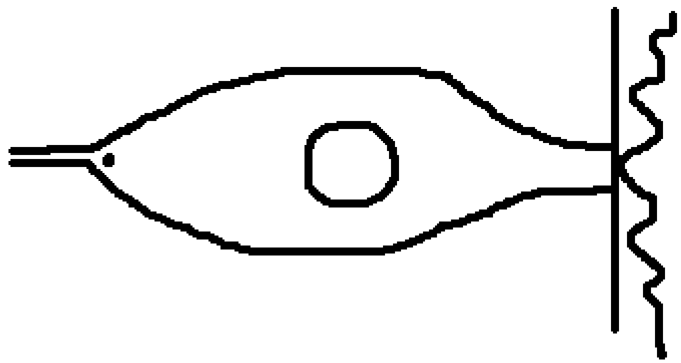
Wilson Loops

- Wilson Loops are a gauge invariant quantity
- With a complete set of Wilson loops, one is able to rebuild all information about a field theory.
- The simplest example of a Wilson loop is the calculation for the Aharonov-Bohm effect.



Aharonov–Bohm Effect

- Consider electrons passing around a solenoid and hitting a detector on the other side.



- From electromagnetism, we have the 4-vector potential

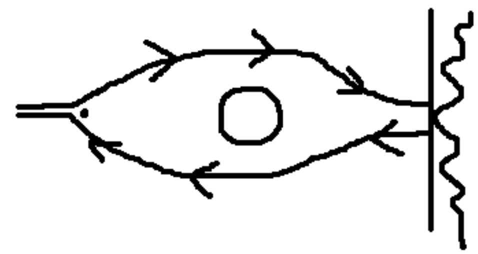
$$A_\mu = (\varphi, A_i)$$

$$B = \nabla \times A \quad E = -\nabla\varphi + \frac{\partial A}{\partial t}$$

Aharonov–Bohm Effect

- Integrating that vector around the contour created by the paths of the electrons, gives a non-zero value, when the solenoid is turned on.

$$\oint A_\mu dx^\mu \neq 0$$

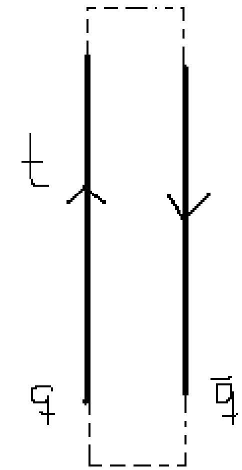


- This is in essence a Wilson loop.

$$\langle W_P \rangle = e^{i \oint_P A_\mu dx^\mu}$$

Wilson Loop Usage

- The picture below illustrates a Wilson loop used to calculate the interaction energy between a quark and an anti-quark.



- The Wilson Loop in this case comes out to have the following form.

$$\langle W_P \rangle \cong e^{-E_{q\bar{q}}T}$$

Wilson Loop Usage

- Wilson loops of the following form indicate a coulomb like force.

$$\langle W_P \rangle \cong e^{-\frac{c}{L}T}$$

- Wilson loops of the following form indicate a confinement like force.

$$\langle W_P \rangle \cong e^{-cLT}$$

- You can see as the distance between the quarks goes to infinity, so does their energy. This is the essence of confinement

Large N field theories and string theory

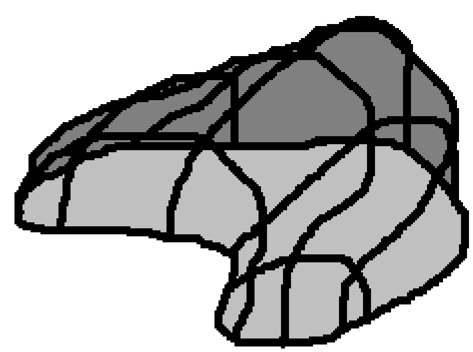
- Wilson loops are relatively easy to calculate perturbatively in field theories with small numbers of force particles
- As N goes to infinity, higher order terms cannot be ignored
- A paper states that Wilson loops can be calculated with the surface area of a minimal area spanning the contour.

$$\langle W_P \rangle \cong e^{-Area}$$

Area Minimization

- The path integral for computing the Wilson Loop can be transformed into a surface integral.
- The surface to be integrated is the surface of least area ending on the path.
- The surface of least area can be strange in the string theory metric.

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$



Calculus of Variations

- We start with an expression for area.

$$A_T = \int A(z, \partial_\mu z)$$

- Then, we allow the path to vary.

$$z = z' + \delta z \qquad A_T = \int A(z' + \delta z, \partial_\mu z' + \partial_\mu \delta z)$$

$$= \int A(z', \partial_\mu z') + \frac{\partial A}{\partial z}\delta z + \frac{\partial A}{\partial(\partial_\mu z)}\partial_\mu(\delta z) + \cdots$$

Calculus of Variations

- Then, the variation in the total area is,

$$\delta A_T = \int \frac{\partial A}{\partial z}\delta z + \frac{\partial A}{\partial(\partial_\mu z)}\partial_\mu(\delta z)$$

- Using integration by parts, we get,

$$\delta A_T = \int \frac{\partial A}{\partial z}\delta z - \partial_\mu\left(\frac{\partial A}{\partial(\partial_\mu z)}\right)\delta z + \left[\frac{\partial A}{\partial(\partial_\mu z)}\,\delta z\right]$$

$$\delta A_T = \int \left(\frac{\partial A}{\partial z} - \partial_\mu\left(\frac{\partial A}{\partial(\partial_\mu z)}\right)\right)\delta z$$

Calculus of Variations

- Thus if we want the variation to go to zero, We have the Euler-Lagrange equations for multiple variables.

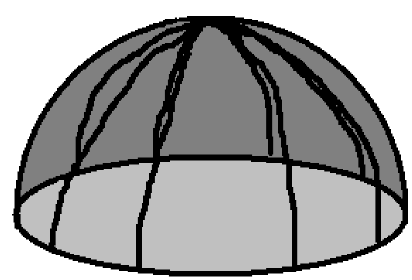
$$\frac{\partial A}{\partial z} - \partial_\mu \left(\frac{\partial A}{\partial(\partial_\mu z)} \right) = 0$$

- This gives the equations for two variables,

$$\frac{\partial A}{\partial z} - \partial_x \left(\frac{\partial A}{\partial(\partial_x z)} \right) - \partial_y \left(\frac{\partial A}{\partial(\partial_y z)} \right) = 0$$

Analytic Solutions

- A few situations have been solved exactly.
- If the boundary of the area to be calculated is a circle, the answer can be proved to be a sphere centered at zero.
- Abstracting only slightly to an ellipse gives you an unsolved problem, and the purpose of my summer here.

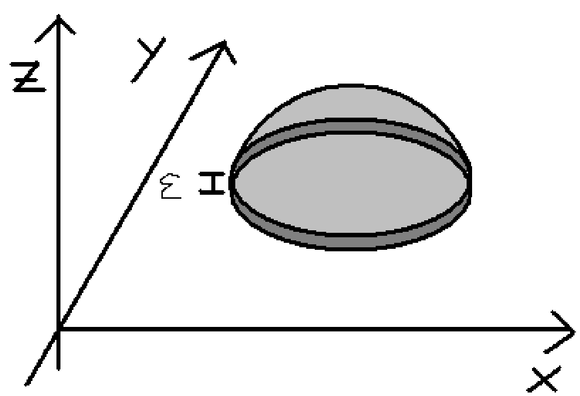


Solving Computationally

- We have first elected to try to solve the problem computationally.
- The first step is to write a program which tries to find such a least surface area, and show that the program gives the correct answer for a circle, a hemisphere.
- After this has been shown, the idea is to run the program on the ellipse to perhaps determine more clearly a form for the solution.

Renormalization

- Due to the metric string theory operates in, the contour is not allowed to be at $z = 0$, but at some $z = \epsilon$.



Renormalization

- The Area for any surface then takes on the form,

$$Area = \frac{P}{\varepsilon} + A_0$$

- Hence, the piece of information we are most interested in, is the A_0.

- This is a form of renormalization. As ϵ -> 0, the Area is infinite, but A_0 stays the same.

Results

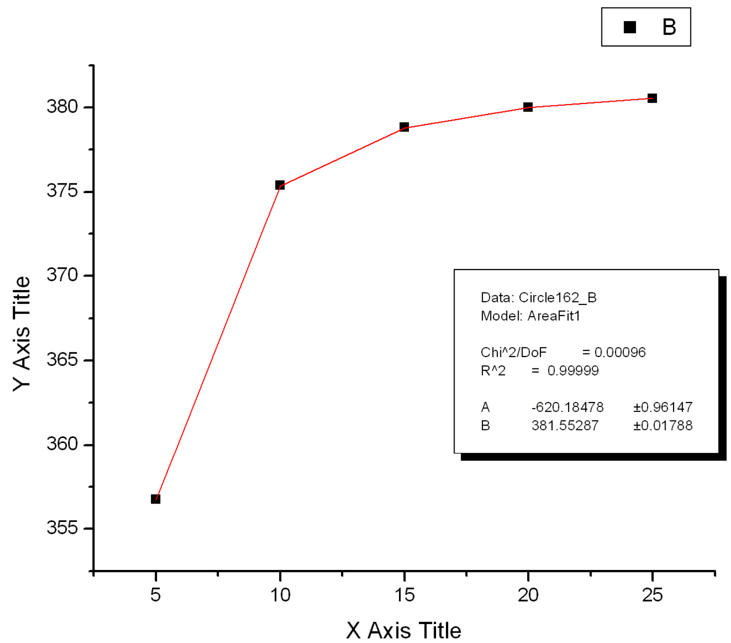
- The following graphs are the results of interpolation of the resultant areas.
- The closest fit for a circle was of the form

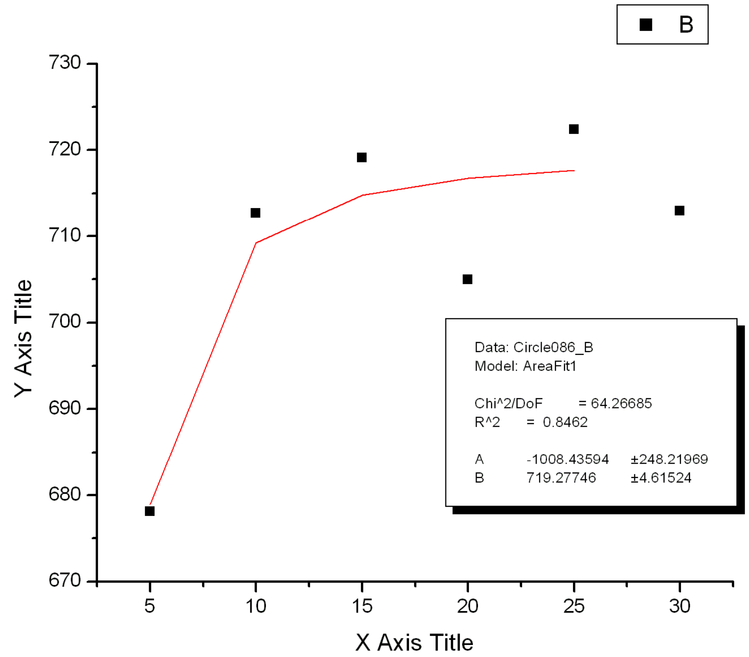
$$Area = A\,f(n) + B \qquad Area = \frac{A}{n^2} + B$$

- Then,treating B as the area for the continuous shape, we apply the above formula to find A_0.
- A similar procedure is applied to different contours, but since most are not solved, we have no way of being sure of the resolution dependence.

Circle Results

- Here are some circle results

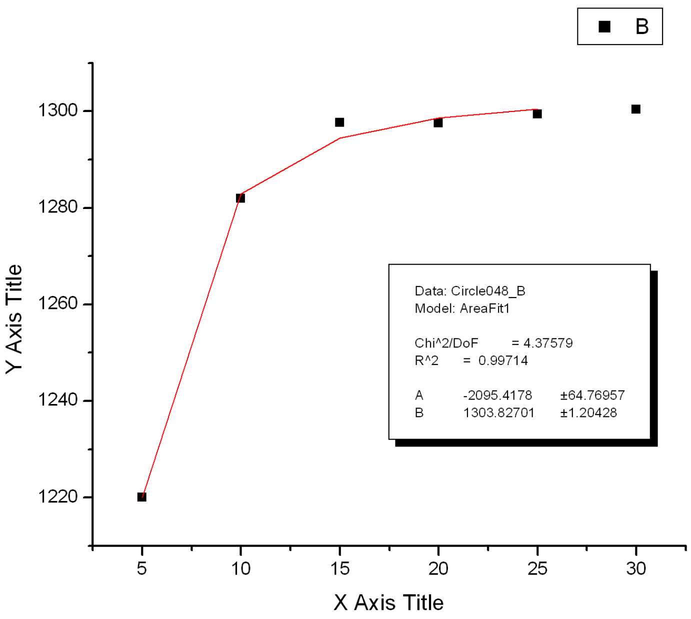


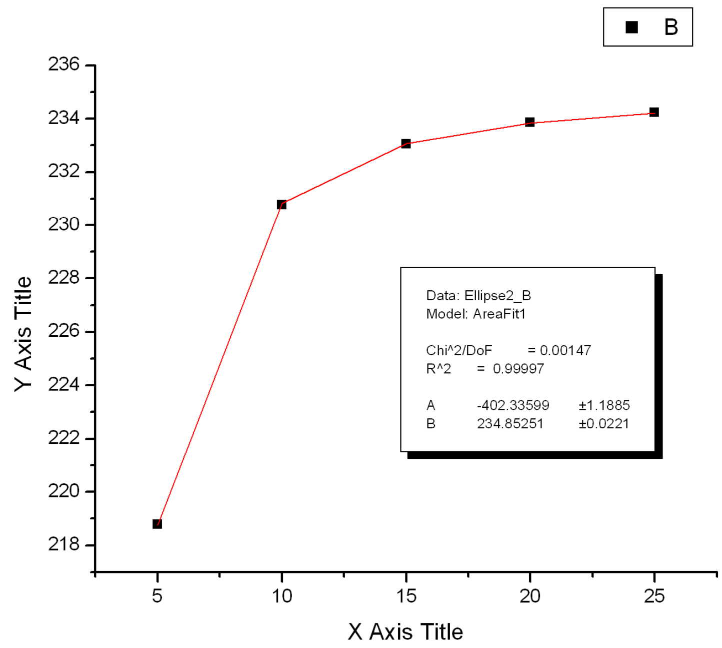


- The Circle had an average A_0 of −6.843. This is within .2 of the correct answer of −2π, or −6.283

Ellipse Results

- Here are some ellipse results

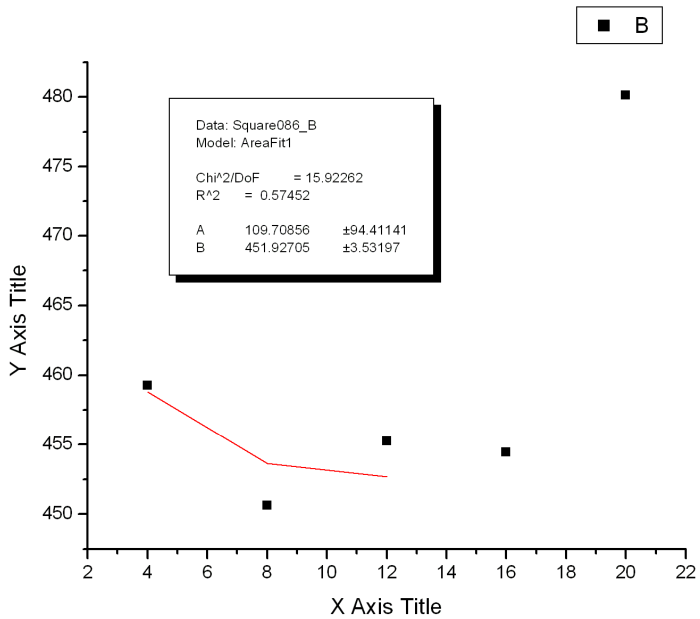


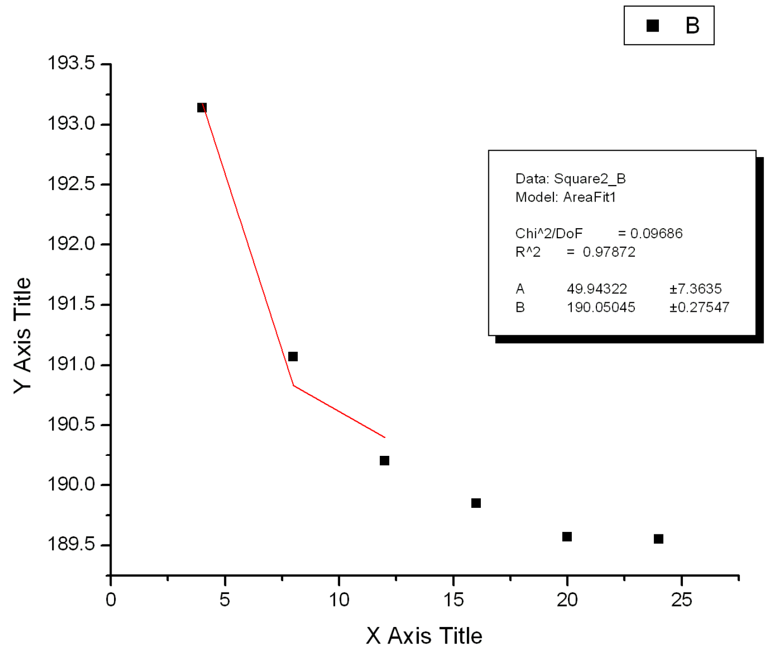


- The ellipse gave an average A_0 of −7.95

Square Results

- Here are some square results





- The square data gave an average A_0 of –33.01 with a wide variance.

Future Goals

- Find the root of the run issues.
- Using clues from the program, try to find an analytic solution for various contours.
- Expand the program's functionality to include concave contours.

Thanks! --- Questions?