

LETTER TO THE EDITOR

Hydrodynamic modes in a trapped strongly interacting Fermi gas of atoms

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Received 9 May 2005, in final form 9 June 2005

Published 4 July 2005

Online at stacks.iop.org/JPhysB/38/L243

Abstract

The zero-temperature properties of a dilute two-component Fermi gas in the Bardeen–Cooper–Schrieffer phase–Bose–Einstein condensate crossover are investigated. On the basis of a generalization of the variational Schwinger method, we construct approximate semi-analytical formulae for collective frequencies of the radial and the axial breathing modes of the Fermi gas under harmonic confinement in the framework of the hydrodynamic theory. It is shown that the method gives nearly exact solutions.

1. Introduction

The newly created ultracold trapped Fermi gases with tunable atomic scattering length [1–20] in the vicinity of a Feshbach resonance offer the possibility of studying highly correlated many-body systems including the crossover from the Bardeen–Cooper–Schrieffer (BCS) phase to the Bose–Einstein condensate (BEC) of molecules. Various investigations based on the hydrodynamic theory have appeared recently [21–31].

The purpose of this letter is to construct simple, semi-analytical and nearly exact formulae for hydrodynamic frequencies. Since the collective frequencies can be measured with high precision, these formulae will provide a simple quantitative tool for the analysis of experimental data in the hydrodynamic regime.

2. Hydrodynamic theory

Our starting point is the quantum hydrodynamic theory [21–23] for a dilute two-component Fermi gas in a trap potential $V_{\text{ext}}(\vec{r}) = (m/2)(\omega_{\perp}^2(x^2 + y^2) + \omega_z^2 z^2)$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V_{\text{ext}} \Psi + V_{\text{xc}} \Psi, \quad (1)$$

where

$$V_{xc}(\vec{r}, t) = \left[\frac{\partial(n\epsilon(n))}{\partial n} \right]_{n=n(\vec{r}, t)},$$

ϵ is the ground state energy per particle of the homogeneous system and n is the density, $n(\vec{r}, t) = |\Psi(\vec{r}, t)|^2$, normalized to the total number of atoms, $\int n(\vec{r}, t) d^3r = N$. It is useful to rewrite equation (1) in the form

$$\frac{\partial n}{\partial t} + \nabla(n\vec{v}) = 0, \quad (2)$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{m} \nabla \left(V_{\text{ext}} + \frac{d(n\epsilon(n))}{dn} + \frac{1}{2}mv^2 - \frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} \right) = 0, \quad (3)$$

where \vec{v} is the velocity field, which for $\Psi = e^{i\phi(\vec{r}, t)} n^{1/2}(\vec{r}, t)$ can be written as $\vec{v} = (\hbar/m)\nabla\phi$.

It can be proved [21] that every solution of equations (2), (3) is a stationary point corresponding to the Lagrangian density

$$\mathcal{L}_0 = \hbar\dot{\phi}n + \frac{\hbar^2}{2m}(\nabla\sqrt{n})^2 + \frac{\hbar^2}{2m}n(\nabla\phi)^2 + \epsilon(n)n + V_{\text{ext}}n. \quad (4)$$

It was shown in [21–23] that for experimental conditions of [14, 16, 17] the quantum pressure term in equations (3) and (4) can be neglected. For the remainder of this letter we will use this hydrodynamic approximation. For the harmonic trap a trial function in the scaling ansatz is taken as [21–23, 26, 28]

$$\phi(\vec{r}, t) = \phi_0(t) + (m/(2\hbar)) \sum_{i=1}^3 \beta_i(t)x_i^2, \quad n(\vec{r}, t) = n_0(x_i/b_i(t))/\kappa(t),$$

where $\kappa(t) = \prod_j b_j$ and the Hamilton principle, $\delta \int dt \int \mathcal{L}_0 d^3r = 0$, gives the following equations for the scaling parameters b_i [21, 22]:

$$\ddot{b}_i + \omega_i^2(t)b_i - \frac{\omega_i^2}{b_i} \frac{\int [n^2 d\epsilon(n)/dn]_{n=n_0(\vec{r})/\kappa(t)} d^3r}{\int [n^2 d\epsilon(n)/dn]_{n=n_0(\vec{r})} d^3r} \kappa(t) = 0. \quad (5)$$

Expanding equation (5) around equilibrium ($b_i = 1$) leads to the following result for the $M = 0$ modes frequencies, $\omega^{(s)}$ in the scaling approximation,

$$\omega_{\pm}^{(s)} = \frac{\omega_{\perp}}{\sqrt{2}} [\eta_s \pm \sqrt{\eta_s^2 - 8\lambda^2(3\zeta_s + 5)}]^{1/2}, \quad (6)$$

where $\eta_s = 4 + 2\zeta_s + 3\lambda^2 + \zeta_s\lambda^2$, $\zeta_s = \int n_0^3 d^2\epsilon/(dn_0^2) d^3r / \int n_0^2 d\epsilon/(dn_0) d^3r$, $\lambda = \omega_z/\omega_{\perp}$, and \pm refer to the transverse and axial mode, respectively.

The hydrodynamic equations after linearization take the form

$$\frac{\partial^2}{\partial t^2} \delta n + \frac{1}{m} \nabla \left(n_0 \nabla \left(\frac{d^2(n_0\epsilon(n_0))}{dn_0^2} \delta n \right) \right) = 0, \quad (7)$$

where $\delta n(\vec{r}, t)$ is the change in the density profile with respect to the equilibrium configuration. If we consider oscillations with time dependence $\delta n \propto \exp(i\omega t)$, equation (7) can be reduced to a Hermitian equation [27]

$$\omega^2 \left[\frac{d^2(n_0\epsilon(n_0))}{dn_0^2} \right]^{-1} |f\rangle = L|f\rangle \quad (8)$$

where $|f\rangle = \frac{d^2(n_0\epsilon(n_0))}{dn_0^2}|\delta n\rangle$, $L = -\frac{1}{m}\nabla n_0\nabla$ and the equilibrium density, n_0 , is given by equation

$$\mu = V_{\text{ext}} + \frac{d(n_0\epsilon(n_0))}{dn_0}, \quad (9)$$

where μ is the chemical potential, in the region where $n_0(\vec{r})$ is positive and $n_0(\vec{r}) = 0$ outside this region.

We note here that $d^2(n_0\epsilon(n_0))/dn_0^2$ is positive, since the sound velocity for the homogeneous case is given by $c^2 = (n_0/m) d^2(n_0\epsilon(n_0))/dn_0^2$.

3. The equation of state

For the negative s-wave scattering length between the two fermionic species, $a < 0$, in the low-density regime, $k_F|a| \ll 1$, the ground state energy per particle, $\epsilon(n)$, is well represented by an expansion in power of $k_F|a|$ [32]

$$\epsilon(n) = 2E_F \left[\frac{3}{10} - \frac{1}{3\pi}k_F|a| + 0.055\,661(k_F|a|)^2 - 0.009\,14(k_F|a|)^3 + \dots \right], \quad (10)$$

where $E_F = \hbar^2 k_F^2 / (2m)$ and $k_F = (3\pi^2 n)^{1/3}$. In the opposite regime, $a \rightarrow -\infty$ (the Bertsch many-body problem, quoted in [33]), $\epsilon(n)$ is proportional to that of the non-interacting Fermi gas

$$\epsilon(n) = (1 + \beta) \frac{3}{10} \frac{\hbar^2 k_F^2}{m}, \quad (11)$$

where a universal parameter β [10] is estimated to be $\beta = -0.56$ [34]. The universal limit [10, 34–37] is valid at least in the case where the width of the Feshbach resonance is large compared to the Fermi energy as in the cases of ${}^6\text{Li}$ and ${}^{40}\text{K}$.

In the $a \rightarrow +0$ limit the system reduces to the dilute Bose gas of dimers

$$\epsilon(n) = E_F(-1/(k_F a)^2 + a_m k_F / (6\pi) + \dots), \quad (12)$$

where a_m is the boson–boson scattering length, $a_m \approx 0.6a$ [38].

A simple interpolation of the form $\epsilon(n) \approx E_F P(k_F a)$ with a smooth function $P(x)$ was considered in several papers. In [21] a [2/2] Padé approximant has been proposed for the function $P(x)$ for the case of negative a ,

$$P(x) = \frac{3}{5} - 2 \frac{\delta_1|x| + \delta_2 x^2}{1 + \delta_3|x| + \delta_4 x^2}, \quad (13)$$

where $\delta_1 = 0.106\,103$, $\delta_2 = 0.187\,515$, $\delta_3 = 2.291\,88$, $\delta_4 = 1.116\,16$. Equation (13) is constructed to reproduce the first four terms of the expansion (10) in the low-density regime and also to exactly reproduce results of the recent Monte Carlo calculations [34], $\beta = -0.56$, in the unitary limit, $k_F a \rightarrow -\infty$.

For the positive a case (the interaction is strong enough to form bound molecules with energy E_{mol}) we have considered in [22] a [2/2] Padé approximant

$$P(x) = \frac{E_{\text{mol}}}{2E_F} + \frac{\alpha_1 x + \alpha_2 x^2}{1 + \alpha_3 x + \alpha_4 x^2}, \quad (14)$$

where parameters α_i are fixed by two continuity conditions at large x , $1/x \rightarrow 0$, and by two continuity conditions at small x , $\alpha_1 = 0.0316621$, $\alpha_2 = 0.0111816$, $\alpha_3 = 0.200149$, and $\alpha_4 = 0.0423545$.

In [39] a Padé approximation has been considered for the chemical potential. Authors of [28] have used a model for $P(x)$, interpolating the Monte Carlo results of [36] across the unitary limit and limiting behaviours for small $|x|$. We note here also the BCS mean-field calculations of [28].

4. The Schwinger variational principle

The Schwinger variational principle (SVP) [40–43] can be generalized to the case of equation (8). Since the entire treatment is based on the equivalence of the SVP and the method of separable representation, we briefly describe this method. Let us consider the symbolic identity

$$L = LL^{-1}L = \sum_{i,j} L|i\rangle\langle i|L^{-1}|j\rangle\langle j|L, \quad (15)$$

where $|i\rangle$ is a complete set. Truncating the summation over the complete set we obtain a separable approximation

$$L^{(q)} = \sum_{i,j}^q L|\chi_i\rangle d_{ij}^{-1} \langle \chi_j|L, \quad (16)$$

where $d_{ij} = \langle \chi_i|L|\chi_j\rangle$.

We note that equation (16) represents an interpolation process, since $L^{(q)}|\chi_i\rangle = L|\chi_i\rangle$ and $\langle \chi_j|L^{(q)} = \langle \chi_j|L$.

Substituting $L^{(q)}$ from equation (16) into equation (8), we obtain

$$\omega^2 \left[\frac{d^2(n_0 \epsilon(n_0))}{dn_0^2} \right]^{-1} |f\rangle = L^{(q)}|f\rangle = \sum_{i,j}^q L|\chi_i\rangle d_{ij}^{-1} \langle \chi_j|L|f\rangle. \quad (17)$$

We seek a solution of equation (17) in the form

$$|f\rangle = \sum_i^q c_i \frac{d^2(n_0 \epsilon(n_0))}{dn_0^2} L|\chi_i\rangle, \quad (18)$$

then c_i are defined from equations

$$\sum_{k=1}^q B_{ik}(\omega^2) c_k = 0, \quad (19)$$

where

$$B_{ik}(\omega^2) = \langle \chi_i | \left(\omega^2 L - L \frac{d^2(n_0 \epsilon(n_0))}{dn_0^2} L \right) | \chi_k \rangle \quad (20)$$

and frequencies ω are determined from the condition of vanishing of the determinant of the matrix $B_{ik}(\omega^2)$:

$$\det B_{ik}(\omega^2) = 0. \quad (21)$$

Defining the Schwinger functional $I_{\text{SVP}}[\chi]$ by

$$I_{\text{SVP}}[\chi] = \frac{\langle \chi | L \frac{d^2(n_0 \epsilon(n_0))}{dn_0^2} L | \chi \rangle}{\langle \chi | L | \chi \rangle}, \quad (22)$$

we get $I_{\text{SVP}}[f] = \omega^2$, where f is the solution of equation (8). Introducing the function $\chi = \sum_{r=1}^q c_r \chi_r$, where c_1, c_2, \dots, c_q are q variable parameters, we see that the functional I_{SVP} is stationary if $\partial I_{\text{SVP}}/\partial c_r = 0$. The latter equations coincide with equation (19), and therefore the approximate solution of equation (8), which is based on the SVP, is equivalent to the exact solution of equation (8) with separable $L^{(q)}$.

Operator L is clearly positive, which means that $\langle u|L|u \rangle \geq 0$ for all u , but is not positive definite, since $\langle u|L|u \rangle = 0$ for some $u \neq 0$. It can easily be seen that

$$\langle u|(L - L^{(q)})|u \rangle \geq 0, \tag{23}$$

for all u . Indeed,

$$J = \left\langle \left(u + \sum_{i=1}^q c_i \chi_i \right) \middle| L \middle| \left(u + \sum_{j=1}^q c_j \chi_j \right) \right\rangle \geq 0$$

for all c_i . We choose the c_i from the conditions $\partial J/\partial c_i = 0$, and then obtain

$$J = \langle u|(L - L^{(q)})|u \rangle \geq 0$$

for all q .

Since the problem is solved by replacing L by $L^{(q)}$ and since the operator $(L - L^{(q)})$ is positive, the SVP leads to the approximate lower bounds for ω up to the second order of $|L - L^{(q)}|$.

For the most interested case of $M = 0$ modes, we can put in equation (16) $q = 2$, $\chi_1 = (x^2 + y^2)$ and $\chi_2 = z^2$, which gives

$$\omega_{\pm}^{\text{SVP}} = \frac{\omega_{\perp}}{\sqrt{2}} \left[\eta_{\text{SVP}} \pm \sqrt{\eta_{\text{SVP}}^2 - 8\lambda^2(9\zeta_{\text{SVP}} - 1)} \right]^{1/2}, \tag{24}$$

where $\eta_{\text{SVP}} = 6\zeta_{\text{SVP}} + \lambda^2(3\zeta_{\text{SVP}} + 1)$, $\zeta_{\text{SVP}} = -\int n_0^2 \tilde{x}^3 \left[\frac{\partial n_0}{\partial \tilde{x}} \right]^{-1} d\tilde{x} / \int n_0 \tilde{x}^4 d\tilde{x}$, $\tilde{x} = \sqrt{x^2 + y^2 + \lambda^2 z^2}$ and \pm signs refer to the transverse and axial mode, respectively.

It is easy to show that equation (24) gives exact solutions for frequencies of the breathing modes for the polytropic equation of state, $\epsilon(n) \approx n^\gamma$. In [24], on the basis of a generalization of the Hylleraas–Undheim method, we have constructed rigorous upper bounds to the collective frequencies for the radial and the axial breathing mode of the Fermi gas under harmonic confinement in the framework of the hydrodynamic theory,

$$\omega_{\pm}^{\text{upper}} = \frac{\omega_{\perp}}{\sqrt{5\zeta_{\text{upper}} - 9}} \left[\eta_{\text{upper}} \pm \sqrt{\eta_{\text{upper}}^2 - 8\lambda^2 \zeta_{\text{upper}} (5\zeta_{\text{upper}} - 9)} \right]^{1/2}, \tag{25}$$

where $\eta_{\text{upper}} = (3 + 4\lambda^2)\zeta_{\text{upper}} - (3 + 6\lambda^2)$, $\zeta_{\text{upper}} = I_0 I_4 / I_2^2$, and $I_l = \int \tilde{x}^l n_0(\tilde{x}) d\tilde{x}$.

We expect that the difference between $\omega_{\pm}^{\text{SVP}}$ and $\omega_{\pm}^{\text{upper}}$ characterizes the error and is not very sensitive to the functional form of $\epsilon(n)$. From table 1 one can see that this difference is order of 10^{-5} . We note that the comparison with the scaling approximation, equation (6), shows that the absolute precision of the scaling approximation is about 10^{-3} that agrees with [31].

In figure 1, we have compared the hydrodynamic predictions for ω_+ with the experimental data [17]. There is a very good agreement with the experimental data [17] near the unitary limit. We note here that two experimental results [17] and [16] (not shown in figure 1) for ω_+ are still about 10% in disagreement with each other, which is not fully understood yet.

To calculate ζ , we have used the very fast converged expansion of [24],

$$n_0(\vec{r}) \approx (1 - \beta V_{\text{ext}}(\vec{r}))^{1/(2-p)} \sum_{i=0}^{l-1} \tilde{c}_i [V_{\text{ext}}(\vec{r})]^i,$$

where parameters β , p and \tilde{c}_i are fixed by requiring that $n_0(\vec{r})$ must satisfy a variational principle $\delta \int n_0 (V_{\text{ext}} + \epsilon(n_0)) d^3r = 0$ with a subsidiary condition $\int n_0 d^3r = N$.

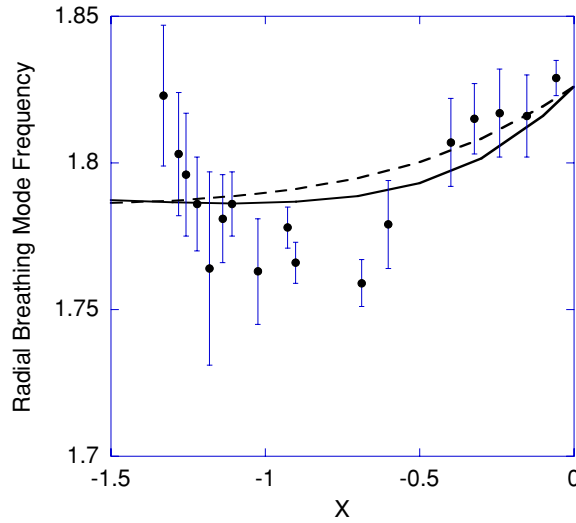


Figure 1. Radial breathing mode frequency ω_+ in the BCS region as a function of the dimensional parameter $X = (N^{1/6}a/a_{ho})^{-1}$ (the solid line). The dashed line represents the scaling approximation, equation (6). The circular dots with error bars are the experimental results given by the Duke University group [17]. Everything is measured in units of ω_\perp .

Table 1. The transverse and axial frequencies in units of ω_\perp and ω_z , respectively, in the BCS region as a function of the dimensional parameter $X = (N^{1/6}a/a_{ho})^{-1}$. The trap parameter λ is assumed to be 0.045 613. The [2/2] Padé approximation of [21, 22] is used for the energy per particle $\epsilon(n)$.

X	ω_+^{upper}	ω_+^{SVP}	ω_-^{upper}	ω_-^{SVP}	ω_+^s	ω_-^s
-0.1	1.8160	1.8160	1.5470	1.5470	1.8193	1.5477
-0.3	1.8015	1.8015	1.5438	1.5438	1.8082	1.5453
-0.5	1.7931	1.7931	1.5419	1.5419	1.8002	1.5435
-0.7	1.7886	1.7886	1.5409	1.5409	1.7947	1.5423
-0.9	1.7867	1.7867	1.5405	1.5405	1.7910	1.5414
-1.1	1.7861	1.7861	1.5403	1.5403	1.7886	1.5409
-1.3	1.7865	1.7865	1.5404	1.5404	1.7871	1.5406
-1.5	1.7873	1.7873	1.5406	1.5406	1.7863	1.5404
-1.7	1.7884	1.7884	1.5409	1.5409	1.7860	1.5403
-2.0	1.7902	1.7902	1.5413	1.5413	1.7861	1.5403

5. Summary

We have generalized the Schwinger variational method for the trapped strongly interacting atoms in the hydrodynamic regime and we have constructed semi-analytical and extremely accurate formulae for hydrodynamic collective frequencies. These formulae are very useful since they provide an easy and simple quantitative tool for the analysis of experimental data for trapped condensed gases without relying on complex and extensive computations.

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